ENHANCED SIX OPERATIONS AND BASE CHANGE THEOREM FOR SHEAVES ON ARTIN STACKS

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Abstract. In this article, we develop a theory of Grothendieck's six operations for lisse-étale sheaves on Artin stacks and prove all expected properties including the Base Change theorem. This extends all previous theories on this subject, including the recent one developed by Laszlo and Olsson. In particular, if we restrict ourselves to constructible sheaves, we obtain the same six operations as Laszlo and Olsson but for more general Artin stacks, with the Base Change isomorphism constructed in the derived category, and without their technical assumptions on the base scheme or on the coefficient rings. Moreover, our theory works for higher Artin stacks as well.

Our method differs from all previous approaches, as we exploit the theory of stable ∞-categories developed by Lurie. These higher categories are viewed as enhancement of usual derived categories. We introduce the ∞-categorical (co)homological descent for Cartesian sheaves and develop several ∞-categorical techniques, which, together with those in our previous article [27], allow us to handle the "homotopy coherence".

Contents

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Introduction

Derived categories of lisse-étale sheaves on Artin stacks, Grothendieck's six operations and the adic formalism have been developed by many authors including [9] (for schemes), [40] (for Deligne–Mumford stacks), [26], [5], [32] and [24, 25]. These theories all have some restrictions. In the most recent and general one [24, 25] by Laszlo and Olsson on Artin stacks, a technical condition was imposed on the base scheme which excludes, for example, the spectra of certain fields[1]. More importantly, the Base Change isomorphism was constructed only on the level of (usual) cohomology sheaves [24, §5]. The Base Change theorem is fundamental in many applications. In the Geometric Langlands Program for example, the theorem has already been used on the level of perverse cohomology. It is thus necessary to construct the Base Change isomorphism not just on the level of cohomology, but also in the derived category. Another limitation of most previous works is that they dealt only with constructible sheaves. When working with morphisms *locally* of finite type, it is desirable to have the six operations for more general lisse-étale sheaves.

In this article, we develop a theory that provides the desired extensions of previous works. Instead of the usual unbounded derived category, we work with its enhancement, which is a stable ∞-category in the sense of Lurie [30, 1.1.1.9]. This makes our approach different from all previous ones. We construct functors and produce relations in the world of ∞-categories, which themselves form an ∞-category. We start by upgrading the known theory of six operations for (disjoint unions of) quasi-compact and separated schemes to ∞-categories. The coherence of the construction is carefully recorded. This enables us to apply ∞-categorical descent to carry over the theory of six operations, including the Base Change theorem, to algebraic spaces, higher Deligne–Mumford stacks and higher Artin stacks.

0.1. Results.

In this section, we will state our results only in the classical setting of Artin stacks on the level of usual derived categories (which are homotopy categories of the derived ∞-categories), among other simplification. We refer the reader to Chapter 6 for a list of complete results for higher Deligne–Mumford stacks and higher Artin stacks, stated on the level of stable ∞-categories.

By an *algebraic space*, we mean a sheaf in the big fppf site satisfying the usual axioms [4, 025Y]: its diagonal is representable (by schemes); and it admits an étale and surjective map from a scheme (in $\mathcal{S}\mathrm{ch}_{\mathcal{U}}$; see §0.5). By an *Artin stack* $\mathcal{X}$, we mean an algebraic stack in the sense of [4, 026O]: it is a stack in (1-)groupoids over $(\mathcal{S}\mathrm{ch}_{\mathcal{U}})_{\mathrm{fppf}}$; its diagonal is representable by algebraic spaces; and it admits a smooth and surjective map from a scheme. In particular, we do not assume that an Artin stack is quasi-separated. Our main results are the construction of the six operations for the derived categories of lisse-étale sheaves on Artin stacks and the expected relations among them. In what follows, Λ is commutative ring (with a unit), or more generally, a ringed diagram in Definition 3.2.1. Let $\mathcal{X}$ be an Artin stack. We denote by $\mathrm{D}(\mathcal{X}_{\text{lis-ét}}, \Lambda)$ the unbounded derived category of $(\mathcal{X}_{\text{lis-ét}}, \Lambda)$-modules, where $\mathcal{X}_{\text{lis-ét}}$ is the lisse-étale topos associated to $\mathcal{X}$. Recall that an $(\mathcal{X}_{\text{lis-ét}}, \Lambda)$-module $\mathscr{F}$ is equivalent to an assignment to each smooth

[1]For example, the field $k(x_1, x_2, \dots)$ obtained by adjoining countably infinitely many variables to an algebraically closed field k in which ℓ is invertible.

morphism $v\colon Y \to \mathcal{X}$ with Y an algebraic space a $(Y_{\text{ét}}, \Lambda)$-module $\mathscr{F}_v$ and to each 2-commutative triangle

$$\begin{tikzcd} Y' \arrow[rr, "f"] \arrow[dr, "v'"'] & \arrow[d, phantom, "\sigma"] & Y \arrow[dl, "v"] \\ & \mathcal{X} & \end{tikzcd}$$

with v, v' smooth and Y, Y' being algebraic spaces, a morphism $\tau_\sigma\colon f^*\mathscr{F}_v \to \mathscr{F}_{v'}$ which is an isomorphism if f is étale such that the collection $\{\tau_\sigma\}$ satisfies a natural cocycle condition [26, 12.2.1]. An $(\mathcal{X}_{\text{lis-ét}}, \Lambda)$-module $\mathscr{F}$ is *Cartesian* if in the above description, *all* morphisms τ_σ are isomorphisms [26, 12.3].

Let $\mathrm{D}_{\mathrm{cart}}(\mathcal{X}_{\text{lis-ét}}, \Lambda)$ be the full subcategory of $\mathrm{D}(\mathcal{X}_{\text{lis-ét}}, \Lambda)$ spanned by complexes whose cohomology sheaves are all Cartesian. If $\mathcal{X}$ is Deligne–Mumford, we have an equivalence $\mathrm{D}_{\mathrm{cart}}(\mathcal{X}_{\text{lis-ét}}, \Lambda) \simeq \mathrm{D}(\mathcal{X}_{\text{ét}}, \Lambda)$.

Let $f\colon \mathcal{Y} \to \mathcal{X}$ be a morphism of Artin stacks. We define operations in §6.1:

$$f^*\colon \mathrm{D}_{\mathrm{cart}}(\mathcal{X}_{\text{lis-ét}}, \Lambda) \to \mathrm{D}_{\mathrm{cart}}(\mathcal{Y}_{\text{lis-ét}}, \Lambda), \quad f_*\colon \mathrm{D}_{\mathrm{cart}}(\mathcal{Y}_{\text{lis-ét}}, \Lambda) \to \mathrm{D}_{\mathrm{cart}}(\mathcal{X}_{\text{lis-ét}}, \Lambda);$$
$$- \otimes_{\mathcal{X}} -\colon \mathrm{D}_{\mathrm{cart}}(\mathcal{X}_{\text{lis-ét}}, \Lambda) \times \mathrm{D}_{\mathrm{cart}}(\mathcal{X}_{\text{lis-ét}}, \Lambda) \to \mathrm{D}_{\mathrm{cart}}(\mathcal{X}_{\text{lis-ét}}, \Lambda),$$
$$\mathbf{Hom}_{\mathcal{X}}\colon \mathrm{D}_{\mathrm{cart}}(\mathcal{X}_{\text{lis-ét}}, \Lambda)^{op} \times \mathrm{D}_{\mathrm{cart}}(\mathcal{X}_{\text{lis-ét}}, \Lambda) \to \mathrm{D}_{\mathrm{cart}}(\mathcal{X}_{\text{lis-ét}}, \Lambda).$$

The pairs (f^*, f_*) and $(- \otimes \mathscr{K}, \mathbf{Hom}(\mathscr{K}, -))$ for every $\mathscr{K} \in \mathrm{D}_{\mathrm{cart}}(\mathcal{X}_{\text{lis-ét}}, \Lambda)$ are pairs of adjoint functors.

We fix a nonempty set L of rational primes. A ring is L-*torsion* [2, IX 1.1] if each element is killed by an integer that is a product of primes in L. An Artin stack $\mathcal{X}$ is L-*coprime* if there exists a morphism $\mathcal{X} \to \operatorname{Spec} \mathbb{Z}[\mathsf{L}^{-1}]$. If $\mathcal{X}$ is L-coprime (resp. Deligne–Mumford), $f\colon \mathcal{Y} \to \mathcal{X}$ is locally of finite type, and Λ is L-torsion (resp. torsion), then we have another pair of adjoint functors:

$$f_!\colon \mathrm{D}_{\mathrm{cart}}(\mathcal{Y}_{\text{lis-ét}}, \Lambda) \to \mathrm{D}_{\mathrm{cart}}(\mathcal{X}_{\text{lis-ét}}, \Lambda), \quad f^!\colon \mathrm{D}_{\mathrm{cart}}(\mathcal{X}_{\text{lis-ét}}, \Lambda) \to \mathrm{D}_{\mathrm{cart}}(\mathcal{Y}_{\text{lis-ét}}, \Lambda).$$

These operations satisfy the following properties, including notably the Base Change theorem.

Theorem 0.1.1 (Base Change, Proposition 6.1.1)**.** *Let Λ be an L-torsion (resp. torsion) ring, and*

$$\begin{tikzcd} \mathcal{W} \arrow[r, "g"] \arrow[d, "q"'] & \mathcal{Z} \arrow[d, "p"] \\ \mathcal{Y} \arrow[r, "f"] & \mathcal{X} \end{tikzcd}$$

be a Cartesian *square of L-coprime Artin stacks (resp. any Deligne–Mumford stacks) where p is locally of finite type. Then we have a natural isomorphism*

$$f^* \circ p_! \simeq q_! \circ g^*\colon \mathrm{D}_{\mathrm{cart}}(\mathcal{Z}_{\text{lis-ét}}, \Lambda) \to \mathrm{D}_{\mathrm{cart}}(\mathcal{Y}_{\text{lis-ét}}, \Lambda)$$

of functors.

Theorem 0.1.2 (Projection Formula, Proposition 6.1.2)**.** *Let Λ be an L-torsion (resp. torsion) ring, and $f\colon \mathcal{Y} \to \mathcal{X}$ be a morphism locally of finite type of L-coprime Artin stacks (resp. any Deligne–Mumford stacks). Then we have a natural isomorphism*

$$f_!(- \otimes_{\mathcal{Y}} f^*-) \simeq (f_!-) \otimes_{\mathcal{X}} -\colon \mathrm{D}_{\mathrm{cart}}(\mathcal{Y}_{\text{lis-ét}}, \Lambda) \times \mathrm{D}_{\mathrm{cart}}(\mathcal{X}_{\text{lis-ét}}, \Lambda) \to \mathrm{D}_{\mathrm{cart}}(\mathcal{X}_{\text{lis-ét}}, \Lambda)$$

of functors.

Corollary 0.1.3 (Künneth Formula, Proposition 6.1.3)**.** *Let Λ be an L-torsion (resp. torsion) ring,*

$$\begin{tikzcd} \mathcal{Y}_1 \arrow[d, "f_1"'] & \mathcal{Y} \arrow[l, "q_1"'] \arrow[r, "q_2"] \arrow[d, "f"] & \mathcal{Y}_2 \arrow[d, "f_2"] \\ \mathcal{X}_1 & \mathcal{X} \arrow[l, "p_1"'] \arrow[r, "p_2"] & \mathcal{X}_2 \end{tikzcd}$$

be a diagram of L-coprime Artin stacks (resp. any Deligne–Mumford stacks) that exhibits $\mathcal{Y}$ as the limit $\mathcal{Y}_1 \times_{\mathcal{X}_1} \mathcal{X} \times_{\mathcal{X}_2} \mathcal{Y}_1$, where f_1 and f_2 are locally of finite type. Then we have a natural isomorphism

$$f_!(q_1^* - \otimes_{\mathcal{Y}} q_2^*-) \simeq (p_1^* f_{1!}-) \otimes_{\mathcal{X}} (p_2^* f_{2!}-)\colon \mathrm{D}_{\mathrm{cart}}(\mathcal{Y}_{1,\text{lis-ét}}, \Lambda) \times \mathrm{D}_{\mathrm{cart}}(\mathcal{Y}_{2,\text{lis-ét}}, \Lambda) \to \mathrm{D}_{\mathrm{cart}}(\mathcal{X}_{\text{lis-ét}}, \Lambda)$$

of functors.

Theorem 0.1.4 (Proposition 6.1.4)**.** *Let Λ be a ring, $f\colon \mathcal{Y} \to \mathcal{X}$ be a morphism of Artin stacks. Then*

(1) *We have a natural isomorphism*
$$f^*(- \otimes_{\mathcal{X}} -) \simeq (f^*-) \otimes_{\mathcal{Y}} (f^*-)\colon \mathrm{D}_{\mathrm{cart}}(\mathcal{X}_{\text{lis-ét}}, \Lambda) \times \mathrm{D}_{\mathrm{cart}}(\mathcal{X}_{\text{lis-ét}}, \Lambda) \to \mathrm{D}_{\mathrm{cart}}(\mathcal{Y}_{\text{lis-ét}}, \Lambda)$$
of functors.

(2) *We have a natural isomorphism*
$$\mathbf{Hom}_{\mathcal{X}}(-, f_*-) \simeq f_*\mathbf{Hom}_{\mathcal{Y}}(f^*-, -)\colon \mathrm{D}_{\mathrm{cart}}(\mathcal{X}_{\text{lis-ét}}, \Lambda)^{op} \times \mathrm{D}_{\mathrm{cart}}(\mathcal{Y}_{\text{lis-ét}}, \Lambda) \to \mathrm{D}_{\mathrm{cart}}(\mathcal{X}_{\text{lis-ét}}, \Lambda)$$
of functors.

(3) *Assume that $\mathcal{X}$ is L-coprime (resp. Deligne–Mumford), Λ is L-torsion (resp. torsion), and f is locally of finite type. We have a natural isomorphism*
$$f^!\mathbf{Hom}_{\mathcal{X}}(-, -) \simeq \mathbf{Hom}_{\mathcal{Y}}(f^*-, f^!-)\colon \mathrm{D}_{\mathrm{cart}}(\mathcal{X}_{\text{lis-ét}}, \Lambda)^{op} \times \mathrm{D}_{\mathrm{cart}}(\mathcal{X}_{\text{lis-ét}}, \Lambda) \to \mathrm{D}_{\mathrm{cart}}(\mathcal{Y}_{\text{lis-ét}}, \Lambda)$$
of functors.

(4) *Assume that $\mathcal{X}$ is L-coprime (resp. Deligne–Mumford), Λ is L-torsion (resp. torsion), and f is locally of finite type. We have a natural isomorphism*
$$f_*\mathbf{Hom}_{\mathcal{Y}}(-, f^!-) \simeq \mathbf{Hom}_{\mathcal{X}}(f_!-, -)\colon \mathrm{D}_{\mathrm{cart}}(\mathcal{Y}_{\text{lis-ét}}, \Lambda)^{op} \times \mathrm{D}_{\mathrm{cart}}(\mathcal{X}_{\text{lis-ét}}, \Lambda) \to \mathrm{D}_{\mathrm{cart}}(\mathcal{X}_{\text{lis-ét}}, \Lambda)$$
of functors.

Theorem 0.1.5 (Smooth (Étale) Base Change, Corollary 6.1.6)**.** *Let Λ of an L-torsion (resp. arbitrary) ring,*
$$\begin{tikzcd} \mathcal{W} \arrow[r, "g"] \arrow[d, "q"'] & \mathcal{Z} \arrow[d, "p"] \\ \mathcal{Y} \arrow[r, "f"] & \mathcal{X} \end{tikzcd}$$
be a Cartesian diagram of L-coprime Artin stacks (resp. any Deligne–Mumford stacks) where p is smooth (resp. étale). Then the natural transformation of functors
$$p^*f_* \to g_*q^*\colon \mathrm{D}_{\mathrm{cart}}(\mathcal{Y}_{\text{lis-ét}}, \Lambda) \to \mathrm{D}_{\mathrm{cart}}(\mathcal{Z}_{\text{lis-ét}}, \Lambda)$$
is a natural isomorphism.

Theorem 0.1.6 (Proposition 6.1.7)**.** *Let Λ be an L-torsion (resp. torsion) ring, and $f\colon \mathcal{Y} \to \mathcal{X}$ be a morphism of L-coprime Artin stacks (resp. any Deligne–Mumford stacks) that is representable by a proper morphism of algebraic spaces. Then we have a natural isomorphism*
$$f_* \simeq f_!\colon \mathrm{D}_{\mathrm{cart}}(\mathcal{Y}_{\text{lis-ét}}, \Lambda) \to \mathrm{D}_{\mathrm{cart}}(\mathcal{X}_{\text{lis-ét}}, \Lambda)$$
of functors.

Theorem 0.1.7 (Proposition 6.1.5)**.** *Let Λ be an L-torsion ring, and $f\colon \mathcal{Y} \to \mathcal{X}$ be a flat morphism locally of finite presentation of L-coprime Artin stacks. Then*

(1) *There is a functorial trace map $\mathrm{Tr}_f\colon \tau^{\geq 0} f_!\Lambda_{\mathcal{Y}}\langle d\rangle = \tau^{\geq 0} f_!(f^*\Lambda_{\mathcal{X}})\langle d\rangle \to \Lambda_{\mathcal{X}}$, where d is an integer larger than or equal to the dimension of every geometric fiber of f; $\Lambda_{\mathcal{X}}$ and $\Lambda_{\mathcal{Y}}$ denote the constant sheaves placed in degree 0; $\langle d\rangle = [2d](d)$ is the composition of the shift by $2d$ and the d-th power of Tate's twist; and $\mathrm{id}_{\mathcal{X}}$ is the identity functor of $\mathrm{D}_{\mathrm{cart}}(\mathcal{X}_{\text{lis-ét}}, \Lambda)$.*

(2) *If f is moreover smooth, the induced natural transformation $u_f\colon f_! \circ f^*\langle \dim f\rangle \to \mathrm{id}_{\mathcal{X}}$ is a counit transformation. In other words, we have a natural isomorphism*
$$f^*\langle \dim f\rangle \simeq f^!\colon \mathrm{D}_{\mathrm{cart}}(\mathcal{X}_{\text{lis-ét}}, \Lambda) \to \mathrm{D}_{\mathrm{cart}}(\mathcal{Y}_{\text{lis-ét}}, \Lambda)$$
of functors.

(3) *If, moreover, $\mathcal{X}$, $\mathcal{Y}$ are Deligne–Mumford stacks and f is locally quasi-finite, then, in (1) and (2), we only need to assume that Λ is torsion and do not need to assume that $\mathcal{X}$ is L-coprime. In this case,* $\dim f = 0$.

Theorem 0.1.8 (Proposition 6.1.8)**.** *Let Λ be a ring, $f\colon \mathcal{Y}\to\mathcal{X}$ be morphism of Artin stacks and $y\colon \mathcal{Y}_0^+\to\mathcal{Y}$ be a smooth surjective morphism. Let $\mathcal{Y}_\bullet^+$ be the Čech nerve of y with the morphism $y_n\colon \mathcal{Y}_n^+\to\mathcal{Y}_{-1}^+=\mathcal{Y}$. Put $f_n=f\circ y_n\colon \mathcal{Y}_n^+\to\mathcal{X}$.*

(1) *For every complex $\mathscr{K}\in \mathrm{D}^{\geq 0}(\mathcal{Y},\Lambda)$, we have a convergent spectral sequence*

$$\mathrm{E}_1^{p,q}=\mathrm{H}^q(f_{p*}y_p^*\mathscr{K})\Rightarrow \mathrm{H}^{p+q}f_*\mathscr{K}.$$

(2) *If $\mathcal{X}$ is L-coprime; Λ is L-torsion, and f is locally of finite type, then for every complex $\mathscr{K}\in \mathrm{D}^{\leq 0}(\mathcal{Y},\Lambda)$, we have a convergent spectral sequence*

$$\tilde{\mathrm{E}}_1^{p,q}=\mathrm{H}^q(f_{-p!}y_{-p}^!\mathscr{K})\Rightarrow \mathrm{H}^{p+q}f_!\mathscr{K}.$$

Let S be either a quasi-excellent scheme or a regular scheme of dimension ≤ 1, that is L-coprime. Let Λ be a Noetherian L-torsion ring. We consider only Artin stacks which are locally of finite type over S. Recall that an $(\mathcal{X}_{\text{lis-ét}},\Lambda)$-module is *constructible* if it is Cartesian and its pullback to every scheme, finite type over S, is constructible in the usual sense. Let $\mathrm{D}_{\mathrm{cons}}(\mathcal{X}_{\text{lis-ét}},\Lambda)$ be the full subcategory of $\mathrm{D}(\mathcal{X}_{\text{lis-ét}},\Lambda)$ spanned by complexes whose cohomology sheaves are constructible. Let $\mathrm{D}^{(+)}_{\mathrm{cons}}(\mathcal{X}_{\text{lis-ét}},\Lambda)$ (resp. $\mathrm{D}^{(-)}_{\mathrm{cons}}(\mathcal{X}_{\text{lis-ét}},\Lambda)$) be the full subcategory of $\mathrm{D}_{\mathrm{cons}}(\mathcal{X}_{\text{lis-ét}},\Lambda)$ spanned by complexes whose cohomology sheaves are locally bounded below (resp. above). The six operations mentioned previously restrict to the following refined ones as in §6.2 (see Lemma 6.2.3 and Proposition 6.2.4 for precise statements):

$$f^*\colon \mathrm{D}_{\mathrm{cons}}(\mathcal{X}_{\text{lis-ét}},\Lambda)\to \mathrm{D}_{\mathrm{cons}}(\mathcal{Y}_{\text{lis-ét}},\Lambda),\quad f^!\colon \mathrm{D}_{\mathrm{cons}}(\mathcal{X}_{\text{lis-ét}},\Lambda)\to \mathrm{D}_{\mathrm{cons}}(\mathcal{Y}_{\text{lis-ét}},\Lambda);$$
$$-\otimes_{\mathcal{X}}-\colon \mathrm{D}^{(-)}_{\mathrm{cons}}(\mathcal{X}_{\text{lis-ét}},\Lambda)\times \mathrm{D}^{(-)}_{\mathrm{cons}}(\mathcal{X}_{\text{lis-ét}},\Lambda)\to \mathrm{D}^{(-)}_{\mathrm{cons}}(\mathcal{X}_{\text{lis-ét}},\Lambda),$$
$$\mathbf{Hom}_{\mathcal{X}}\colon \mathrm{D}^{(-)}_{\mathrm{cons}}(\mathcal{X}_{\text{lis-ét}},\Lambda)^{op}\times \mathrm{D}^{(+)}_{\mathrm{cons}}(\mathcal{X}_{\text{lis-ét}},\Lambda)\to \mathrm{D}^{(+)}_{\mathrm{cons}}(\mathcal{X}_{\text{lis-ét}},\Lambda).$$

If f is *quasi-compact and quasi-separated*, then we have

$$f_*\colon \mathrm{D}^{(+)}_{\mathrm{cons}}(\mathcal{Y}_{\text{lis-ét}},\Lambda)\to \mathrm{D}^{(+)}_{\mathrm{cons}}(\mathcal{X}_{\text{lis-ét}},\Lambda),\quad f_!\colon \mathrm{D}^{(-)}_{\mathrm{cons}}(\mathcal{Y}_{\text{lis-ét}},\Lambda)\to \mathrm{D}^{(-)}_{\mathrm{cons}}(\mathcal{X}_{\text{lis-ét}},\Lambda).$$

We will also show that when the base scheme, the coefficient ring, and the morphism f are all in the range of [24], our operations for constructible complexes are compatible with those of Laszlo–Olsson on the level of usual derived categories. In particular, we prove the Base Change theorem in its full generality, which was left open in [24].

In the subsequent article [28], we will define adic complexes and extend all the previous results to adic complexes. Let (Ξ,Λ) be a partially ordered diagram of coefficient rings, that is, Ξ is a partially ordered set and Λ is a functor from Ξ^{op} to the category of commutative rings (with units). For example, we can fix a prime ℓ and consider the projective system

$$\cdots\to\mathbb{Z}/\ell^{n+1}\mathbb{Z}\to\mathbb{Z}/\ell^n\mathbb{Z}\to\cdots\to\mathbb{Z}/\ell\mathbb{Z},$$

where the transition maps are natural projections. Inside the category $\mathrm{D}_{\mathrm{cart}}(\mathcal{X}^{\Xi}_{\text{lis-ét}},\Lambda)$, there is a full subcategory $\mathrm{D}(\mathcal{X}^{\Xi}_{\text{lis-ét}},\Lambda)_{\mathrm{adic}}$ spanned by (Ξ,Λ)-*adic complexes*. The inclusion admits a right adjoint

$$\mathrm{D}_{\mathrm{cart}}(\mathcal{X}^{\Xi}_{\text{lis-ét}},\Lambda)\to \mathrm{D}(\mathcal{X}^{\Xi}_{\text{lis-ét}},\Lambda)_{\mathrm{adic}}$$

which exhibits $\mathrm{D}(\mathcal{X}^{\Xi}_{\text{lis-ét}},\Lambda)_{\mathrm{adic}}$ as a *colocalization* of $\mathrm{D}_{\mathrm{cart}}(\mathcal{X}^{\Xi}_{\text{lis-ét}},\Lambda)$. We will also define constructible adic complexes and prove analogues of the theorems listed previously.

0.2. **Why ∞-categories?** The ∞-categories in this article refer to the ones studied by A. Joyal [21,22] (where they are called *quasi-categories*), J. Lurie [29], et al. Namely, an ∞-category is a simplicial set satisfying lifting properties of inner horn inclusions [29, 1.1.2.4]. In particular, they are models for $(\infty,1)$-categories, that is, higher categories whose n-morphisms are invertible for $n\geq 2$. For readers who are not familiar with this language, we recommend [17] for a brief introduction of Lurie's theory [29], [30], etc. There are also other models for $(\infty,1)$-categories such as topological categories, simplicial categories, complete Segal spaces, Segal categories, model categories, and, in a looser sense, differential graded (DG) categories and A_∞-categories. We address two questions in this section. First, why do we need $(\infty,1)$-categories instead of (usual) derived categories? Second, why do we choose this particular model of $(\infty,1)$-categories?

To answer these questions, let us fix an Artin stack $\mathcal{X}$ and an atlas $u\colon X \to \mathcal{X}$, that is, a smooth and surjective morphism with X an algebraic space. We denote by $\mathrm{Mod}(\mathcal{X}_{\text{lis-ét}},\Lambda)$ (resp. $\mathrm{Mod}(X_{\text{ét}},\Lambda)$) the category of $(\mathcal{X}_{\text{lis-ét}},\Lambda)$-modules (resp. $(X_{\text{ét}},\Lambda)$-modules) which is a Grothendieck abelian category. Let $p_\alpha\colon X\times_{\mathcal{X}} X\to X$ ($\alpha=1,2$) be the two projections. We know that if $\mathscr{F}\in\mathrm{Mod}(\mathcal{X}_{\text{lis-ét}},\Lambda)$ is Cartesian, then there is a natural isomorphism $\sigma\colon p_1^*u^*\mathscr{F}\xrightarrow{\sim}p_2^*u^*\mathscr{F}$ satisfying a cocycle condition. Conversely, an object $\mathscr{G}\in\mathrm{Mod}(X_{\text{ét}},\Lambda)$ such that there exists an isomorphism $\sigma\colon p_1^*\mathscr{G}\xrightarrow{\sim}p_2^*\mathscr{G}$ satisfying the same cocycle condition is isomorphic to $u^*\mathscr{F}$ for some $\mathscr{F}\in\mathrm{Mod}(\mathcal{X}_{\text{lis-ét}},\Lambda)$. This descent property can be described in the following formal way. Let $\mathrm{Mod}_{\mathrm{cart}}(\mathcal{X}_{\text{lis-ét}},\Lambda)$ be the full subcategory of $\mathrm{Mod}(\mathcal{X}_{\text{lis-ét}},\Lambda)$ spanned by Cartesian sheaves. Then it is the (2-)limit of the following diagram

$$\mathrm{Mod}(X_{\text{ét}},\Lambda)\underset{p_2^*}{\overset{p_1^*}{\rightrightarrows}}\mathrm{Mod}((X\times_{\mathcal{X}}X)_{\text{ét}},\Lambda)\;\substack{\rightarrow\\\rightarrow\\\rightarrow}\;\mathrm{Mod}((X\times_{\mathcal{X}}X\times_{\mathcal{X}}X)_{\text{ét}},\Lambda)$$

in the $(2,1)$-category of abelian categories[2]. Therefore, to study $\mathrm{Mod}_{\mathrm{cart}}(\mathcal{X}_{\text{lis-ét}},\Lambda)$, we only need to study $\mathrm{Mod}(X_{\text{ét}},\Lambda)$ for (all) algebraic spaces X in a "2-coherent way", that is, we need to track down all the information of natural isomorphisms (2-cells). Such 2-coherence is not more complicated than the one in Grothendieck's theory of descent [18].

One may want to apply the same idea to derived categories. The problem is that the descent property mentioned previously, in its naïve sense, does not hold anymore, since otherwise the classifying stack $\mathrm{B}\mathbb{G}_{\mathrm{m}}$ over an algebraically closed field will have finite cohomological dimension which is incorrect. In fact, when forming derived categories, we throw away too much information on the coherence of homotopy equivalences or quasi-isomorphisms, which causes the failure of such descent. A descent theory in a weaker sense, known as cohomological descent [2, V bis] and due to Deligne, does exist partially on the level of objects. It is one of the main techniques used in Olsson [32] and Laszlo–Olsson [24] for the definition of the six operations on Artin stacks in certain cases. However, it has the following restrictions. First, Deligne's cohomological descent is valid only for complexes bounded below. Although a theory of cohomological descent for unbounded complexes was developed in [24], it comes at the price of imposing further finiteness conditions and restricting to constructible complexes. Second, relevant spectral sequences suggest that cohomological descent cannot be used directly to define !-pushforward.

A more natural solution can be reached once the derived categories are "enhanced". Roughly speaking (see Proposition 5.3.4 for the precise statement), if we write $X_n=X\times_{\mathcal{X}}\cdots\times_{\mathcal{X}}X$ ($(n+1)$-fold), then $\mathcal{D}_{\mathrm{cart}}(\mathcal{X}_{\text{lis-ét}},\Lambda)$ is naturally equivalent to the limit of following cosimplicial diagram

$$\mathcal{D}(X_{0,\text{ét}},\Lambda)\underset{p_2^*}{\overset{p_1^*}{\rightrightarrows}}\mathcal{D}(X_{1,\text{ét}},\Lambda)\;\substack{\rightarrow\\\rightarrow\\\rightarrow}\;\mathcal{D}(X_{2,\text{ét}},\Lambda)\;\substack{\rightarrow\\\rightarrow\\\rightarrow\\\rightarrow}\;\cdots$$

in a suitable ∞-category of *closed symmetric monoidal presentable stable* ∞*-categories*. This is completely parallel to the descent property for module categories. Here $\mathcal{D}_{\mathrm{cart}}(\mathcal{X}_{\text{lis-ét}},\Lambda)$ (resp. $\mathcal{D}(X_{n,\text{ét}},\Lambda)$) is a closed symmetric monoidal presentable stable ∞-category which serves as the enhancement of $\mathrm{D}_{\mathrm{cart}}(\mathcal{X}_{\text{lis-ét}},\Lambda)$ (resp. $\mathrm{D}(X_{n,\text{ét}},\Lambda)$). Strictly speaking, the previous diagram is incomplete in the sense that we do not mark all the higher cells in the diagram, that is, all natural isomorphisms of functors, "isomorphisms among natural isomorphisms", etc. In fact, there is an infinite hierarchy of (homotopy) equivalences hidden behind the limit of the previous diagram, not just the 2-level hierarchy in the classical case. To deal with such kind of "homotopy coherence" is the major difficulty of the work, that is, we need to find a way to encode all such hierarchy *simultaneously* in order to make the idea of descent work. In other words, we need to work in the *totality* of all ∞-categories of concern.

It is possible that such a descent theory (and other relevant higher-categorical techniques introduced below) can be realized by using other models for higher categories. We have chosen the theory developed by Lurie in [29], [30] for its elegance and availability. Precisely, we will use the techniques of the (marked) straightening/unstraightening construction, Adjoint Functor Theorem, and the ∞-categorical Barr–Beck Theorem. Based on Lurie's theory, we develop further ∞-categorical techniques to treat the homotopy-coherence problem mentioned as above. These techniques would enable us to, for example,

[2]A $(2,1)$-category is a 2-category in which all 2-cells are invertible.

- take partial adjoints along given directions (§1.4);
- find a coherent way to decompose morphisms ([27, §4]);
- gluing data from Cartesian diagrams to general ones ([27, §6]);
- make a coherent choice of descent data (§4.2).

In the next section, we will have a chance to explain some of them.

During the preparation of this article, Gaitsgory [13] studied operations for ind-coherent sheaves on DG schemes and derived stacks in the framework of ∞-categories. Our work bears some similarity to his. We would like to point out however that he ignored homotopy-theoretical issues (in the same sense of homotopy coherence), for example, in the proof of [13, 6.1.9], which is a key step for the entire construction. Meanwhile, a sizable portion (Chapter 1 and [27]) of our work is devoted to developing general techniques to handle homotopy coherence.

We would also like to remark that Lurie's theory has already been used, for example, in [6] to study quasi-coherent sheaves on certain (derived) stacks with many applications. This work, which studies lisse-étale sheaves, is another manifestation of the power of Lurie's theory.

0.3. **What do we need to enhance?** In the previous section, we mentioned the enhancement of a single derived category. It is a stable ∞-category (which can be thought of as an ∞-categorical version of a triangulated category) $\mathcal{D}_{\mathrm{cart}}(\mathcal{X}_{\text{lis-ét}},\Lambda)$ (resp. $\mathcal{D}(X_{\text{ét}},\Lambda)$ for X an algebraic space) whose homotopy category (which is an ordinary category) is naturally equivalent to $\mathrm{D}_{\mathrm{cart}}(\mathcal{X}_{\text{lis-ét}},\Lambda)$ (resp. $\mathrm{D}(X_{\text{ét}},\Lambda)$). The enhancement of operations is understood in the similar way. For example, the enhancement of $*$-pullback for $f\colon \mathcal{Y}\to\mathcal{X}$ should be an exact functor

$$f^*\colon \mathcal{D}_{\mathrm{cart}}(\mathcal{X}_{\text{lis-ét}},\Lambda)\to \mathcal{D}_{\mathrm{cart}}(\mathcal{Y}_{\text{lis-ét}},\Lambda) \tag{0.1}$$

such that the induced functor

$$\mathrm{h}f^*\colon \mathrm{D}_{\mathrm{cart}}(\mathcal{X}_{\text{lis-ét}},\Lambda)\to \mathrm{D}_{\mathrm{cart}}(\mathcal{Y}_{\text{lis-ét}},\Lambda)$$

is the $*$-pullback functor of usual derived categories.

However, such enhancement is not enough for us to do descent. The reason is that we need to put all schemes and then algebraic spaces together. Let us denote by $\mathcal{S}\mathrm{ch}^{\mathrm{qc.sep}}$ the category of disjoint unions of quasi-compact and separated schemes. The enhancement of $*$-pullback for schemes in the strong sense is a functor:

$$\mathcal{S}\mathrm{ch}^{\mathrm{qc.sep}}\overset{\Lambda}{\mathrm{EO}}{}^*\colon \mathrm{N}(\mathcal{S}\mathrm{ch}^{\mathrm{qc.sep}})^{op}\to \mathcal{P}\mathrm{r}^{\mathrm{L}}_{\mathrm{st}} \tag{0.2}$$

where N denotes the nerve functor (see the definition preceding [29, 1.1.2.2]) and $\mathcal{P}\mathrm{r}^{\mathrm{L}}_{\mathrm{st}}$ is certain ∞-category of presentable stable ∞-categories, which will be specified later. Then (0.1) is just the image of the 1-cell $f\colon \mathcal{Y}\to\mathcal{X}$ if f is in $\mathcal{S}\mathrm{ch}^{\mathrm{qc.sep}}$. The construction of (0.2) (and its right adjoint which is the enhancement of $*$-pushforward) is not hard, with the help of the general construction in [30]. The difficulty arises in the enhancement of !-pushforward. Namely, we need to construct a functor:

$$\mathcal{S}\mathrm{ch}^{\mathrm{qc.sep}}\overset{\Lambda}{\mathrm{EO}}{}_!\colon \mathrm{N}(\mathcal{S}\mathrm{ch}^{\mathrm{qc.sep}})_F\to \mathcal{P}\mathrm{r}^{\mathrm{L}}_{\mathrm{st}}, \tag{0.3}$$

where $\mathrm{N}(\mathcal{S}\mathrm{ch}^{\mathrm{qc.sep}})_F$ is the subcategory of $\mathrm{N}(\mathcal{S}\mathrm{ch}^{\mathrm{qc.sep}})$ only allowing morphisms that are locally of finite type. The basic idea is similar to the classical approach: using Nagata compactification theorem. The problem is the following: for a morphism $f\colon Y\to X$ in $\mathcal{S}\mathrm{ch}^{\mathrm{qc.sep}}$, locally of finite type, we need to choose (non-canonically!) a relative compactification

$$\begin{tikzcd} Y \arrow[r, "i"] \arrow[d, "f"'] & \overline{Y} \arrow[d, "\overline{f}"] \\ X & \coprod_I X, \arrow[l, "p"'] \end{tikzcd}$$

that is, i is an open immersion and $\overline{f}$ is proper, and define $f_! = p_!\circ\overline{f}_*\circ i_!$ (in the derived sense). It turns out that the resulting functor of usual derived categories, up to natural isomorphism, is independent of such choice. First, we need to upgrade such natural isomorphism to the level of ∞-categories (which is usually called natural equivalence). Second and more importantly, we need to "remember" such natural

equivalences for all different compactifications, and even "equivalences among natural equivalences". We immediately find ourselves in the same scenario of an infinity hierarchy of homotopy equivalences again. For handling this kind of homotopy coherence, we use a technique called *multisimplicial descent* in [27, §4], which can be viewed as an ∞-categorical generalization of [2, XVII 3.3].

This is not the end of the story since our goal is to prove all expected relations among six operations. To use the same idea of descent, we need to "enhance" not just operations, but also relations as well. To simplify the discussion, let us temporarily ignore the two binary operations ($\otimes$ and $\mathbf{Hom}$) and consider how to enhance the "Base Change theorem" which essentially involves $*$-pullback and !-pushforward. We define a simplicial set $\delta^*_{2,\{2\}}\mathrm{N}(\mathcal{S}\mathrm{ch}^{\mathrm{qc.sep}})^{\mathrm{cart}}_{F,A}$ in the following way:

- The 0-cells are objects X of $\mathcal{S}\mathrm{ch}^{\mathrm{qc.sep}}$.
- The 1-cells are *Cartesian* diagrams

(0.4)
$$\begin{tikzcd} X_{01} \arrow[r, "g"] \arrow[d, "q"'] & X_{00} \arrow[d, "p"] \\ X_{11} \arrow[r, "f"] & X_{10} \end{tikzcd}$$

  with p locally of finite type, whose source is X_{00} and target is X_{11}.
- The higher cells are defined in a similar way.

Note that this is *not* an ∞-category. Assuming that Λ is torsion, the enhancement of the Base Change theorem (for $\mathcal{S}\mathrm{ch}^{\mathrm{qc.sep}}$) is a functor

$$_{\mathcal{S}\mathrm{ch}^{\mathrm{qc.sep}}}^{\Lambda}\mathrm{EO}^*_! \colon \delta^*_{2,\{2\}}\mathrm{N}(\mathcal{S}\mathrm{ch}^{\mathrm{qc.sep}})^{\mathrm{cart}}_{F,A} \to \mathcal{P}\mathrm{r}^{\mathrm{L}}_{\mathrm{st}}$$

such that it sends the 1-cell

$$\begin{tikzcd} X_{00} \arrow[r, "\mathrm{id}"] \arrow[d] & X_{00} \arrow[d, "p"] \\ X_{11} \arrow[r, "\mathrm{id}"] & X_{11} \end{tikzcd} \qquad \left(\text{resp.}\quad \begin{tikzcd} X_{11} \arrow[r] \arrow[d, "\mathrm{id}"'] & X_{00} \arrow[d, "\mathrm{id}"] \\ X_{11} \arrow[r, "f"] & X_{00} \end{tikzcd}\right)$$

to $p_! \colon \mathcal{D}(X_{00,\text{ét}},\Lambda) \to \mathcal{D}(X_{11,\text{ét}},\Lambda)$ (resp. $f^* \colon \mathcal{D}(X_{11,\text{ét}},\Lambda) \to \mathcal{D}(X_{00,\text{ét}},\Lambda)$). The upshot is that the image of the 1-cell (0.4) is a functor $\mathcal{D}(X_{11,\text{ét}},\Lambda) \to \mathcal{D}(X_{00,\text{ét}},\Lambda)$ which is naturally equivalent to both $f^* \circ p_!$ and $q_! \circ g^*$. In other words, this functor has already encoded the Base Change theorem (for $\mathcal{S}\mathrm{ch}^{\mathrm{qc.sep}}$) in a homotopy coherent way. This allows us to apply the descent method to construct the enhancement of the Base Change theorem for Artin stacks, which itself includes the enhancement of the four operations f^*, f_*, $f_!$ and $f^!$ by restriction and adjunction. To deal with the homotopy coherence involved in the construction of $_{\mathcal{S}\mathrm{ch}^{\mathrm{qc.sep}}}^{\Lambda}\mathrm{EO}^*_!$, we use another technique called *Cartesian gluing* in [27, §6], which can be viewed as an ∞-categorical variant of [39, §§6, 7].

We hope the discussion so far explains the meaning of enhancement to some degree. The actual enhancement (3.5) constructed in the article is more complicated than the ones mentioned previously, since we need to include also the information of binary operations, the projection formula and extension of scalars.

0.4. **Structure of the article.** The main body of the article is divided into seven chapters. Chapter 1 is a collection of preliminaries on ∞-categories, including the technique of partial adjoints (§1.4) and the introduction of an ∞-operad $\mathcal{P}\mathrm{f}^{\otimes}$ which will be used to encode the projection formula coherently. Chapter 2 is the starting point of the theory, where we construct enhanced operations for ringed topoi. The first two chapters do not involve algebraic geometry.

In Chapter 3, we construct the enhanced operation map for schemes in the category $\mathcal{S}\mathrm{ch}^{\mathrm{qc.sep}}$. The enhanced operation map encodes even more information than the enhancement of the Base Change theorem we mentioned in §0.3. We also prove several properties of the map that are crucial for later constructions.

In Chapter 4, we develop an abstract program which we name DESCENT. The program allows us to extend the existing theory to a larger category. It will be run recursively from schemes to algebraic

spaces, then to Artin stacks, and eventually to higher Artin or Deligne–Mumford stacks. The detailed running process is described in Chapter 5. There, we also prove certain compatibility between our theory and existing ones.

In Chapter 6, we write down the resulting six operations for the most general situations and summarize their properties. We also develop a theory of constructible complexes, based on finiteness results of Deligne [3, Th. finitude] and Gabber [33]. Finally, we show that our theory is compatible with Laszlo-Olsson's work [24].

For more detailed descriptions of the individual chapters, we refer to the beginning of these chapters.

We assume that the reader has some knowledge of Lurie's theory of ∞-categories, especially Chapters 1 through 5 of [29], and Chapters 1, 2 and 6 of [30]. In particular, we assume that the reader is familiar with basic concepts of simplicial sets [29, A.2.7]. However, an effort has been made to provide precise references for notations, concepts, constructions, and results used in this article, (at least) at their first appearance.

0.5. **Conventions and notations.**

- All rings are assumed to be commutative with unity.

For *set-theoretical issues*:

- We fix two (Grothendieck) universes $\mathcal{U}$ and $\mathcal{V}$ such that $\mathcal{U}$ belongs to $\mathcal{V}$. The adjective *small* means $\mathcal{U}$-small. In particular, Grothendieck abelian categories and presentable ∞-categories are relative to $\mathcal{U}$. A topos means a $\mathcal{U}$-topos.
- All rings are assumed to be $\mathcal{U}$-small. We denote by $\mathcal{R}\mathrm{ing}$ the category of rings in $\mathcal{U}$. By the usual abuse of language, we call $\mathcal{R}\mathrm{ing}$ the category of $\mathcal{U}$-small rings.
- All schemes are assumed to be $\mathcal{U}$-small. We denote by $\mathcal{S}\mathrm{ch}$ the category of schemes belonging to $\mathcal{U}$ and by $\mathcal{S}\mathrm{ch}^{\mathrm{aff}}$ the full subcategory consisting of affine schemes belonging to $\mathcal{U}$. We have an equivalence of categories $\mathrm{Spec}\colon (\mathcal{R}\mathrm{ing})^{op} \to \mathcal{S}\mathrm{ch}^{\mathrm{aff}}$. The big fppf site on $\mathcal{S}\mathrm{ch}^{\mathrm{aff}}$ is not a $\mathcal{U}$-site, so that we need to consider prestacks with values in $\mathcal{V}$. More precisely, for $\mathcal{W} = \mathcal{U}$ or $\mathcal{V}$, let $\mathcal{S}_{\mathcal{W}}$ [29, 1.2.16.1] is the ∞-category of spaces in $\mathcal{W}$. We define the ∞-category of prestacks to be $\mathrm{Fun}(\mathrm{N}(\mathcal{S}\mathrm{ch}^{\mathrm{aff}})^{op}, \mathcal{S}_{\mathcal{V}})$ [29, 1.2.7.2]. However, a (higher) Artin stack is assumed to be contained in the essential image of the full subcategory $\mathrm{Fun}(\mathrm{N}(\mathcal{S}\mathrm{ch}^{\mathrm{aff}})^{op}, \mathcal{S}_{\mathcal{U}})$. See §5.4 for more details.

  The (small) étale site of an algebraic scheme and the lisse-étale site of an Artin stack are equivalent to $\mathcal{U}$-small sites. In particular, they are $\mathcal{U}$-sites.
- For every $\mathcal{V}$-small set I, we denote by $\mathcal{S}\mathrm{et}^I_\Delta$ the category of I-simplicial sets in $\mathcal{V}$. See also variants in §1.3. We denote by $\mathcal{C}\mathrm{at}_\infty$ the (non $\mathcal{V}$-small) ∞-category of ∞-categories in $\mathcal{V}$ [29, 3.0.0.1][3]. (Multi)simplicial sets and ∞-categories are usually tacitly assumed to be $\mathcal{V}$-small.

For *lower categories*:

- Unless otherwise specified, a category will be understood as an ordinary category. A $(2,1)$-category $\mathcal{C}$ is a (strict) 2-category in which all 2-cells are invertible, or, equivalently, a category enriched in the category of groupoids. We regard $\mathcal{C}$ as a simplicial category by taking $\mathrm{N}(\mathrm{Map}_{\mathcal{C}}(X,Y))$ for all objects X and Y of $\mathcal{C}$.
- Let $\mathcal{C}, \mathcal{D}$ be two categories. We denote by $\mathrm{Fun}(\mathcal{C}, \mathcal{D})$ the *category of functors* from $\mathcal{C}$ to $\mathcal{D}$, whose objects are functors and morphisms are natural transformations.
- Let $\mathcal{A}$ be an additive category. We denote by $\mathrm{Ch}(\mathcal{A})$ the category of cochain complexes of $\mathcal{A}$.
- Recall that a *partially ordered set* P is an (ordinary) category such that there is at most one arrow (usual denoted as $\leq$) between each pair of objects. For every element $p \in P$, we identify the overcategory $P_{/p}$ (resp. undercategory $P_{p/}$) with the full partially ordered subset of P consisting of elements $\leq p$ (resp. $\geq p$). In particular, for $p, p' \in P$, $P_{p//p'}$ is identified with the full partially ordered subset of P consisting of elements both $\geq p$ and $\leq p'$, which is empty unless $p \leq p'$.
- Let $[n]$ be the ordered set $\{0, \dots, n\}$ for $n \geq 0$ and $[-1] = \emptyset$. Let us recall the *category of combinatorial simplices* $\mathbf{\Delta}$ (resp. $\mathbf{\Delta}^{\leq n}$, $\mathbf{\Delta}_+$, $\mathbf{\Delta}_+^{\leq n}$). Its objects are the linearly ordered sets $[i]$ for $i \geq 0$ (resp. $0 \leq i \leq n$, $i \geq -1$, $-1 \leq i \leq n$) and its morphisms are given by (nonstrictly)

[3]In [29], $\mathcal{C}\mathrm{at}_\infty$ denotes the category of small ∞-categories. Thus our $\mathcal{C}\mathrm{at}_\infty$ corresponds more closely to the notation $\widehat{\mathcal{C}\mathrm{at}}_\infty$ in [29, 3.0.0.5], where the extension of universes is tacit.

order-preserving maps. In particular, for every $n \geq 0$ and $0 \leq k \leq n$, we have the face map $d^n_k \colon [n-1] \to [n]$ that is the unique injective map with k not in the image; and the degeneration map $s^n_k \colon [n+1] \to [n]$ that is the unique surjective map such that $s^n_k(k+1) = s^n_k(k)$.

For *higher categories*:

- As we have mentioned, the word ∞-*category* refers to the one defined in [29, 1.1.2.4]. Throughout the article, an effort has been made to keep our notations consistent with those in [29] and [30].
- For $\mathcal{C}$ a category, a $(2,1)$-category, a simplicial category, or an ∞-category, we denote by $\mathrm{id}_{\mathcal{C}}$ the identity functor of $\mathcal{C}$. We denote by $\mathrm{N}(\mathcal{C})$ the (simplicial) nerve of a (simplicial) category $\mathcal{C}$ [29, 1.1.5.5]. We identify $\mathrm{Ar}(\mathcal{C})$ (the set of arrows of $\mathcal{C}$) with $\mathrm{N}(\mathcal{C})_1$ (the set of 1-cells of $\mathcal{C}$) if $\mathcal{C}$ is a category. Usually, we will not distinguish between $\mathrm{N}(\mathcal{C}^{op})$ and $\mathrm{N}(\mathcal{C})^{op}$ for $\mathcal{C}$ a category, a $(2,1)$-category or a simplicial category.
- We denote the homotopy category [29, 1.1.3.2, 1.2.3.1] of an ∞-category $\mathcal{C}$ by $\mathrm{h}\mathcal{C}$ and we view it as an ordinary category. In other words, we ignore the $\mathcal{H}$-enrichment of $\mathrm{h}\mathcal{C}$.
- Let $\mathcal{C}$ be an ∞-category and $c^\bullet \colon \mathrm{N}(\mathbf{\Delta}) \to \mathcal{C}$ (resp. $c_\bullet \colon \mathrm{N}(\mathbf{\Delta}^{op}) \to \mathcal{C}$) be a cosimplicial (resp. simplicial) object of $\mathcal{C}$. Then the limit [29, 1.2.13.4] $\varprojlim(c^\bullet)$ (resp. colimit or geometric realization $\varinjlim(c_\bullet)$), if it exists, is denoted by $\varprojlim_{n\in\mathbf{\Delta}} c^n$ (resp. $\varinjlim_{n\in\mathbf{\Delta}^{op}} c_n$). It is viewed as an object (up to equivalences parameterized by a contractible Kan complex) of $\mathcal{C}$.
- Let $\mathcal{C}$ be an (∞-)category, $\mathcal{C}' \subseteq \mathcal{C}$ be a full subcategory. We say a morphism $f \colon y \to x$ in $\mathcal{C}$ is *representable in* $\mathcal{C}'$ if for every Cartesian diagram [29, 4.4.2]

$$\begin{tikzcd} w \arrow[r] \arrow[d] & z \arrow[d] \\ y \arrow[r, "f"] & x \end{tikzcd}$$

  such that z is an object of $\mathcal{C}'$, w is equivalent to an object of $\mathcal{C}'$.
- We refer the reader to the beginning of [29, 2.3.3] for the terminology *homotopic relative to A over S*. We say f and f' are *homotopic over S* (resp. *homotopic relative to A*) if $A = \emptyset$ (resp. $S = *$).
- Recall that $\mathcal{C}\mathrm{at}_\infty$ is the ∞-category of $\mathcal{V}$-small ∞-categories. In [29, 5.5.3.1], the subcategories $\mathcal{P}\mathrm{r}^{\mathrm{L}}, \mathcal{P}\mathrm{r}^{\mathrm{R}} \subseteq \mathcal{C}\mathrm{at}_\infty$ are defined[4]. We define subcategories $\mathcal{P}\mathrm{r}^{\mathrm{L}}_{\mathrm{st}}, \mathcal{P}\mathrm{r}^{\mathrm{R}}_{\mathrm{st}} \subseteq \mathcal{C}\mathrm{at}_\infty$ as follows:
  - The objects of both $\mathcal{P}\mathrm{r}^{\mathrm{L}}_{\mathrm{st}}$ and $\mathcal{P}\mathrm{r}^{\mathrm{R}}_{\mathrm{st}}$ are the $\mathcal{U}$-presentable stable ∞-categories in $\mathcal{V}$ [29, 5.5.0.1], [30, 1.1.1.9].
  - A functor $F \colon \mathcal{C} \to \mathcal{D}$ of presentable stable ∞-categories is a morphism of $\mathcal{P}\mathrm{r}^{\mathrm{L}}_{\mathrm{st}}$ if and only if F preserves small colimits, or, equivalently, F is a left adjoint functor [29, 5.2.2.1, 5.5.2.9 (1)].
  - A functor $G \colon \mathcal{C} \to \mathcal{D}$ of presentable stable ∞-categories is a morphism of $\mathcal{P}\mathrm{r}^{\mathrm{R}}_{\mathrm{st}}$ if and only if G is accessible and preserves small limits, or, equivalently, G is a right adjoint functor [29, 5.5.2.9 (2)].

  We adopt the notations of [29, 5.2.6.1]: for ∞-categories $\mathcal{C}$ and $\mathcal{D}$, we denote by $\mathrm{Fun}^{\mathrm{L}}(\mathcal{C},\mathcal{D})$ (resp. $\mathrm{Fun}^{\mathrm{R}}(\mathcal{C},\mathcal{D})$) the full subcategory of $\mathrm{Fun}(\mathcal{C},\mathcal{D})$ [29, 1.2.7.2] spanned by left (resp. right) adjoint functors. Small limits exist in $\mathcal{C}\mathrm{at}_\infty$, $\mathcal{P}\mathrm{r}^{\mathrm{L}}$, $\mathcal{P}\mathrm{r}^{\mathrm{R}}$, $\mathcal{P}\mathrm{r}^{\mathrm{L}}_{\mathrm{st}}$ and $\mathcal{P}\mathrm{r}^{\mathrm{R}}_{\mathrm{st}}$. Such limits are preserved by the natural inclusions $\mathcal{P}\mathrm{r}^{\mathrm{L}}_{\mathrm{st}} \subseteq \mathcal{P}\mathrm{r}^{\mathrm{L}} \subseteq \mathcal{C}\mathrm{at}_\infty$ and $\mathcal{P}\mathrm{r}^{\mathrm{R}}_{\mathrm{st}} \subseteq \mathcal{P}\mathrm{r}^{\mathrm{R}} \subseteq \mathcal{C}\mathrm{at}_\infty$ by [29, 5.5.3.13, 5.5.3.18] and [30, 1.1.4.4].
- For the simplicial model category $\mathcal{S}\mathrm{et}^+_\Delta$ of marked simplicial sets in $\mathcal{V}$ [29, 3.1.0.2] with respect to the Cartesian model structure [29, 3.1.3.7, 3.1.4.4] , we fix a *fibrant replacement simplicial functor* $\mathrm{Fibr} \colon \mathcal{S}\mathrm{et}^+_\Delta \to (\mathcal{S}\mathrm{et}^+_\Delta)^\circ$ via the Small Object Argument [29, A.1.2.5, A.1.2.6]. By construction, it commutes with finite products. If $\mathcal{C}$ is a $\mathcal{V}$-small simplicial category [29, 1.1.4.1], we let $\mathrm{Fibr}^{\mathcal{C}} \colon (\mathcal{S}\mathrm{et}^+_\Delta)^{\mathcal{C}} \to ((\mathcal{S}\mathrm{et}^+_\Delta)^\circ)^{\mathcal{C}} \subseteq (\mathcal{S}\mathrm{et}^+_\Delta)^{\mathcal{C}}$ be the induced fibrant replacement simplicial functor with respect to the projective model structure [29, A.3.3.1].

[4]Under our convention, the objects of $\mathcal{P}\mathrm{r}^{\mathrm{L}}$ and $\mathcal{P}\mathrm{r}^{\mathrm{R}}$ are the $\mathcal{U}$-presentable ∞-categories in $\mathcal{V}$.

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1. Preliminaries on ∞-categories

This chapter is a collection of preliminaries on ∞-categories. In §1.1, we record some basic lemmas. In §1.2, we recall a key lemma and its variant established in [27], which will be subsequently used in this article. In §1.3, we recall the definitions of multisimplicial sets and multi-marked simplicial sets from [27]. In §1.4, we develop a method of taking partial adjoints, namely, taking adjoint functors along given directions. This will be used to construct the initial enhanced operation map for schemes. In §1.5, we collect some general facts about symmetric monoidal ∞-categories, including a closure property of closed symmetric monoidal presentable ∞-categories. We also introduce an ∞-operad $\mathcal{P}\mathrm{f}^{\otimes}$ to coherently encode the projection formula in the construction of enhanced operation maps in latter chapters.

1.1. **Elementary lemmas.** Let us start with the following lemma, which shows up as [31, 2.4.6]. We include a proof for the convenience of the reader.

Lemma 1.1.1. *Let $\mathcal{C}$ be a nonempty ∞-category which admits product of two objects. Then the geometric realization $|\mathcal{C}|$ is contractible.*

Proof. Fix an object X of $\mathcal{C}$ and a functor $\mathcal{C} \to \mathcal{C}$ sending Y to $X \times Y$. The projections $X \times Y \to X$ and $X \times Y \to Y$ define functors $h, h' \colon \Delta^1 \times \mathcal{C} \to \mathcal{C}$ such that

- $h \mid \Delta^{\{0\}} \times \mathcal{C} = h' \mid \Delta^{\{0\}} \times \mathcal{C}$;
- $h \mid \Delta^{\{1\}} \times \mathcal{C}$ is the constant functor of value X;
- $h' \mid \Delta^{\{1\}} \times \mathcal{C} = \mathrm{id}_{\mathcal{C}}$.

Then $|h|$ and $|h'|$ provide a homotopy between $\mathrm{id}_{|\mathcal{C}|}$ and the constant map of value X. □

The following is a variant of the Adjoint Functor Theorem [29, 5.5.2.9].

Lemma 1.1.2. *Let $F \colon \mathcal{C} \to \mathcal{D}$ be a functor between presentable ∞-categories. Let $\mathrm{h}F \colon \mathrm{h}\mathcal{C} \to \mathrm{h}\mathcal{D}$ be the functor of (unenriched) homotopy categories.*

1. *The functor F has a right adjoint if and only if it preserves pushouts and $\mathrm{h}F$ has a right adjoint.*
2. *The functor F has a left adjoint if and only if it is accessible and preserves pullbacks and $\mathrm{h}F$ has a left adjoint.*

Proof. The necessity follows from [29, 5.2.2.9]. The sufficiency in (1) follows from the fact that small colimits can be constructed out of pushouts and small coproducts [29, 4.4.2.7] and preservation of small coproducts can be tested on $\mathrm{h}F$. The sufficiency in (2) follows from dual statements. □

We will apply the above lemma in the following form.

Lemma 1.1.3. *Let $F \colon \mathcal{C} \to \mathcal{D}$ be a functor between presentable stable ∞-categories. Let $\mathrm{h}F \colon \mathrm{h}\mathcal{C} \to \mathrm{h}\mathcal{D}$ be the functor of (unenriched) homotopy categories. Then*

1. *The functor F admits a right adjoint if and only if $\mathrm{h}F$ is a triangulated functor and admits a right adjoint.*
2. *The functor F admits a left adjoint if F admits a right adjoint and $\mathrm{h}F$ admits a left adjoint.*

Proof. By [30, 1.2.4.12], a functor G between stable ∞-categories is exact if and only if $\mathrm{h}G$ is triangulated. The lemma then follows from Lemma 1.1.2 and [30, 1.1.4.1]. □

1.2. **The Key Lemma.** In this section we recall the key lemma from [27, §2]. It is crucial for many constructions in both articles.

Let K be a simplicial set. Recall that the *category of simplices of* K [29, 6.1.2.5], denoted by $\mathbf{\Delta}_{/K}$, is the strict fiber product $\mathbf{\Delta}\times_{\mathcal{S}\mathrm{et}_\Delta}(\mathcal{S}\mathrm{et}_\Delta)_{/K}$. An object of $\mathbf{\Delta}_{/K}$ is a pair (J,σ), where $J\in\Delta$ and $\sigma\in\mathrm{Hom}_{\mathcal{S}\mathrm{et}_\Delta}(\Delta^J,K)$. A morphism $(J,\sigma)\to(J',\sigma')$ is a map $d\colon\Delta^J\to\Delta^{J'}$ such that $\sigma=\sigma'\circ d$. Let M be a marked simplicial set. We define an object $\mathrm{Map}[K,M]$ of the diagram category $(\mathcal{S}\mathrm{et}_\Delta)^{(\mathbf{\Delta}_{/K})^{op}}$ by

$$\mathrm{Map}[K,M](J,\sigma)=\mathrm{Map}^\sharp((\Delta^J)^\flat,M),$$

for every $(J,\sigma)\in\mathbf{\Delta}_{/K}$. If $M=\mathcal{C}^\natural$ for some ∞-category $\mathcal{C}$, we let $\mathrm{Map}[K,\mathcal{C}]=\mathrm{Map}[K,M]$. Then $\mathrm{Map}[K,\mathcal{C}](J,\sigma)$ is the largest Kan complex [29, 1.2.5.3] contained in $\mathrm{Fun}(\Delta^J,\mathcal{C})$. A morphism d in $\mathbf{\Delta}_{/K}$ goes to the natural restriction map Res^d of Kan mapping complexes. The right adjoint of the diagonal functor $\mathcal{S}\mathrm{et}_\Delta\to(\mathcal{S}\mathrm{et}_\Delta)^{(\mathbf{\Delta}_{/K})^{op}}$ is the global section functor

$$\Gamma\colon(\mathcal{S}\mathrm{et}_\Delta)^{(\mathbf{\Delta}_{/K})^{op}}\to\mathcal{S}\mathrm{et}_\Delta,\quad \Gamma(\mathcal{N})_q=\mathrm{Hom}_{(\mathcal{S}\mathrm{et}_\Delta)^{(\mathbf{\Delta}_{/K})^{op}}}(\Delta^q_K,\mathcal{N}),$$

where $\Delta^q_K\colon(\mathbf{\Delta}_{/K})^{op}\to\mathcal{S}\mathrm{et}_\Delta$ is the constant functor of value Δ^q. We have

$$\Gamma(\mathrm{Map}[K,\mathcal{C}])=\mathrm{Map}^\sharp(K^\flat,\mathcal{C}^\natural).$$

If $g\colon K'\to K$ is a map, composition with the functor $\mathbf{\Delta}_{/K'}\to\mathbf{\Delta}_{/K}$ induced by g defines a functor $g^*\colon(\mathcal{S}\mathrm{et}_\Delta)^{(\mathbf{\Delta}_{/K})^{op}}\to(\mathcal{S}\mathrm{et}_\Delta)^{(\mathbf{\Delta}_{/K'})^{op}}$. We have $g^*\mathrm{Map}[K,M]=\mathrm{Map}[K',M]$.

Let $\Phi\colon\mathcal{M}\to\mathcal{N}$ be a morphism of $(\mathcal{S}\mathrm{et}_\Delta)^{(\mathbf{\Delta}_{/K})^{op}}$. We denote by $\Gamma_\Phi(\mathcal{N})\subseteq\Gamma(\mathcal{N})$ the simplicial subset which is the union of the image of $\Gamma(\Phi')\colon\Gamma(\mathcal{M}')\to\Gamma(\mathcal{N})$ for all decompositions

$$\mathcal{M}\hookrightarrow\mathcal{M}'\xrightarrow{\Phi'}\mathcal{N}$$

of Φ such that $\mathcal{M}(\sigma)\hookrightarrow\mathcal{M}'(\sigma)$ is anodyne [29, 2.0.0.3] for all $\sigma\in\mathbf{\Delta}_{/K}$. The map $\Gamma(\Phi)\colon\Gamma(\mathcal{M})\to\Gamma(\mathcal{N})$ factors through $\Gamma_\Phi(\mathcal{N})$. For every map $g\colon K'\to K$, the canonical map $\Gamma(\mathcal{N})\to\Gamma(g^*\mathcal{N})$ sends $\Gamma_\Phi(\mathcal{N})$ to $\Gamma_{g^*\Phi}(g^*\mathcal{N})$. For every morphism $\psi\colon\mathcal{N}\to\mathcal{N}'$ in $(\mathcal{S}\mathrm{et}_\Delta)^{(\mathbf{\Delta}_{/K})^{op}}$, the map $\Gamma(\psi)\colon\Gamma(\mathcal{N})\to\Gamma(\mathcal{N}')$ induces a map

$$\Gamma_\Phi(\psi)\colon\Gamma_\Phi(\mathcal{N})\to\Gamma_{\psi\circ\Phi}(\mathcal{N}').$$

For every element $a\in\Gamma(\mathcal{N}')_0$ (resp. $a\in\Gamma_{\psi\circ\Phi}(\mathcal{N}')_0$), we denote by $\Gamma^\psi_a(\mathcal{N})$ (resp. $\Gamma^\psi_{\Phi,a}(\mathcal{N})$) the fiber of $\Gamma(\psi)$ (resp. $\Gamma_\Phi(\psi)$) at a. We omit ψ from the notation when no confusion arises.

We can now state the key lemma.

Lemma 1.2.1 ([27, 2.2])**.** *Let $f\colon Z\to T$ be a fibration in $\mathcal{S}\mathrm{et}^+_\Delta$ with respect to the Cartesian model structure, K be a simplicial set, $a\colon K^\flat\to T$ be a map, $\mathcal{N}\in(\mathcal{S}\mathrm{et}_\Delta)^{(\mathbf{\Delta}_{/K})^{op}}$ be such that $\mathcal{N}(\sigma)$ is weakly contractible for all $\sigma\in\mathbf{\Delta}_{/K}$. Consider morphisms $\Phi\colon\mathcal{N}\to\mathrm{Map}[K,Z]$ in $(\mathcal{S}\mathrm{et}_\Delta)^{(\mathbf{\Delta}_{/K})^{op}}$ such that $\mathrm{Map}[K,f]\circ\Phi$ factors through $a\colon\Delta^0_K\to\mathrm{Map}[K,T]$.*

1. *For every such Φ, $\Gamma_{\Phi,a}(\mathrm{Map}[K,Z])$ is a weakly contractible Kan complex.*
2. *Let $\Phi,\Phi'\colon\mathcal{N}\to\mathrm{Map}[K,Z]$ be two such morphisms. Assume that Φ and Φ' are homotopic with respect to a. Then $\Gamma_{\Phi,a}(\mathrm{Map}[K,Z])$ and $\Gamma_{\Phi',a}(\mathrm{Map}[K,Z])$ lie in the same connected component of $\Gamma_a(\mathrm{Map}[K,Z])$.*

Here, by saying Φ and Φ' are homotopic with respect to a, we mean that there exists a morphism $H\colon\Delta^1_K\times\mathcal{N}\to\mathrm{Map}[K,Z]$ in $(\mathcal{S}\mathrm{et}_\Delta)^{(\mathbf{\Delta}_{/K})^{op}}$ such that $H\mid\Delta^{\{0\}}_K\times\mathcal{N}=\Phi$, $H\mid\Delta^{\{1\}}_K\times\mathcal{N}=\Phi'$ and $\mathrm{Map}[K,f]\circ H$ is the identity on a.

Lemma 1.2.1 has the following consequence.

Lemma 1.2.2 ([27, 2.4])**.** *Let K be a simplicial set, $\mathcal{C}$ be an ∞-category, $i\colon A\hookrightarrow B$ be a monomorphism of simplicial sets, $f\colon\mathrm{Fun}(B,\mathcal{C})\to\mathrm{Fun}(A,\mathcal{C})$ be the morphism induced by i. Let $\mathcal{N}$ be an object of $(\mathcal{S}\mathrm{et}_\Delta)^{(\mathbf{\Delta}_{/K})^{op}}$ such that $\mathcal{N}(\sigma)$ is weakly contractible for all σ, and $\Phi\colon\mathcal{N}\to\mathrm{Map}[K,\mathrm{Fun}(B,\mathcal{C})]$ be a morphism such that $\mathrm{Map}[K,f]\circ\Phi$ factors through $a\colon\Delta^0_K\to\mathrm{Map}[K,\mathrm{Fun}(A,\mathcal{C})]$. Then there exists $b\colon\Delta^0_K\to\mathrm{Map}[K,\mathrm{Fun}(B,\mathcal{C})]$ lifting a, such that for every map $g\colon K'\to K$ and every section ν of $g^*\mathcal{N}$, $b\circ g$ and $g^*\Phi\circ\nu\colon K'\to\mathrm{Fun}(B,\mathcal{C})$ are homotopic over $\mathrm{Fun}(A,\mathcal{C})$. Here, in $b\circ g$, we regard b as a map $K\to\mathrm{Fun}(B,\mathcal{C})$.*

1.3. **Multisimplicial sets.** We recall the definitions of multisimplicial sets and multi-marked simplicial sets from [27, §3]. Let I, J be $\mathcal{V}$-small sets.

Definition 1.3.1 (Multisimplicial set). An *I-simplicial set* is a presheaf on the category $\mathbf{\Delta}^I := \mathrm{Fun}(I, \mathbf{\Delta})$. We denote by $\mathcal{S}\mathrm{et}^I_\Delta$ the category of I-simplicial sets.

We denote by $\Delta^{n_i|i\in I}$ the I-simplicial set which, as a presheaf on $\mathbf{\Delta}^I$, is represented by $([n_i])_{i\in I}$. For an I-simplicial set S, we denote by $S_{n_i|i\in I}$ the value of S at $(n_i)_{i\in I} \in \mathbf{\Delta}^I$. A $(n_i)_{i\in I}$*-cell* of an I-simplicial set S is an element of $S_{n_i|i\in I}$. By Yoneda's lemma, there is a canonical bijection of the set $S_{n_i|i\in I}$ and the set of maps from $\Delta^{n_i|i\in I}$ to S.

Let $k \geq 0$ be an integer. A *k-simplicial set* is a $\{1, \dots, k\}$-simplicial set, namely a presheaf S on the category $\mathbf{\Delta}^k := \mathbf{\Delta}^{\{1,\dots,k\}} = \mathbf{\Delta} \times \cdots \times \mathbf{\Delta}$ (k copies).

Let $J \subseteq I$. Composition with the partial opposite functor $\mathbf{\Delta}^I \to \mathbf{\Delta}^I$ sending $(\dots, S_{j'}, \dots, S_j, \dots)$ to $(\dots, S_{j'}, \dots, S_j^{op}, \dots)$ (taking *op* for S_j when $j \in J$) defines a functor $\mathrm{op}^I_J \colon \mathcal{S}\mathrm{et}^I_\Delta \to \mathcal{S}\mathrm{et}^I_\Delta$. We define $\Delta^{n_i|i\in I}_J = \mathrm{op}^I_J \Delta^{n_i|i\in I}$. Although $\Delta^{n_i|i\in I}_J$ is isomorphic to $\Delta^{n_i|i\in I}$, it will be useful in specifying the variance of many constructions. When $I = \{1, \dots, k\}$, we use the notations op^k_J and $\Delta^{n_1,\dots,n_k}_J$.

Let $f \colon J \to I$ be a map of sets. Composition with f defines a functor $\mathbf{\Delta}^f \colon \mathbf{\Delta}^I \to \mathbf{\Delta}^J$. Composition with $\mathbf{\Delta}^f$ induces a functor $(\mathbf{\Delta}^f)^* \colon \mathcal{S}\mathrm{et}^J_\Delta \to \mathcal{S}\mathrm{et}^I_\Delta$. It has a right adjoint $(\mathbf{\Delta}^f)_* \colon \mathcal{S}\mathrm{et}^I_\Delta \to \mathcal{S}\mathrm{et}^J_\Delta$. If $f \colon J \to I$ is a map with $J = \{1, \dots, k'\}$ (resp. and $I = \{1, \dots, k\}$), we will write $\epsilon^I_{f(1)\cdots f(k')}$ (resp. $\epsilon^k_{f(1)\cdots f(k')}$) for $(\mathbf{\Delta}^f)_*$. In particular, $(\epsilon^k_j K)_n = K_{0\cdots n\cdots 0}$, where n is at the j-th position and all other indices are 0.

Let $f \colon I \to \{1\}$. Then $\delta_I := \mathbf{\Delta}^f \colon \mathbf{\Delta} \to \mathbf{\Delta}^I$ is the diagonal map. Composition with δ_I induces a functor $\delta_I^* = (\mathbf{\Delta}^f)^*$ satisfying

$$\delta_I^* \Delta^{n_i|i\in I}_J = \left(\prod_{i\in I-J} \Delta^{n_i}\right) \times \left(\prod_{j\in J} (\Delta^{n_j})^{op}\right) =: \Delta^{[n_i]_{i\in I}}_J.$$

When $J = \emptyset$, we simply write $\Delta^{[n_i]_{i\in I}}$ for $\Delta^{[n_i]_{i\in I}}_\emptyset = \prod_{i\in I} \Delta^{n_i}$. A right adjoint $\delta^I_* \colon \mathcal{S}\mathrm{et}_\Delta \to \mathcal{S}\mathrm{et}^I_\Delta$ of δ_I^* can be described as follows. A $(n_i)_{i\in I}$-cell of $\delta^I_* X$ is given by a map $\Delta^{[n_i]_{i\in I}} \to X$. For $J \subseteq I$, the twisted diagonal functor $\delta^*_{I,J} = \delta^*_I \circ \mathrm{op}^I_J \colon \mathcal{S}\mathrm{et}^I_\Delta \to \mathcal{S}\mathrm{et}_\Delta$ admits the right adjoint $\delta^{I,J}_* = \mathrm{op}^I_J \circ \delta^I_* \colon \mathcal{S}\mathrm{et}_\Delta \to \mathcal{S}\mathrm{et}^I_\Delta$. For a simplicial set X, a $(n_i)_{i\in I}$-cell of $\delta^{I,J}_* X$ consists of a map $\Delta^{[n_i]_{i\in I}}_J \to X$. When $J = \emptyset$, op^I_J is the identity functor so that $\delta^*_{I,\emptyset} = \delta^*_I$, $\delta^{I,\emptyset}_* = \delta^I_*$. When $I = \{1, \dots, k\}$, we write k instead of I in the previous notations. In particular,

$$\delta^*_k = \delta^*_I \colon \mathcal{S}\mathrm{et}^k_\Delta \to \mathcal{S}\mathrm{et}_\Delta, \quad (\delta^*_k X)_n = X_{n\dots n},$$

satisfying $\delta^*_k \Delta^{n_1,\dots,n_k}_J = \Delta^{[n_1,\dots,n_k]}_J$. Moreover, according to our notations, $\delta^k_* = \epsilon^1_{1\cdots 1}$, where the index 1 is repeated k times.

We define a bifunctor

$$\boxtimes \colon \mathcal{S}\mathrm{et}^I_\Delta \times \mathcal{S}\mathrm{et}^J_\Delta \to \mathcal{S}\mathrm{et}^{I \coprod J}_\Delta$$

by the formula $S \boxtimes S' = (\mathbf{\Delta}^{\iota_I})^* S \times (\mathbf{\Delta}^{\iota_J})^* S'$, where $\iota_I \colon I \hookrightarrow I \coprod J$, $\iota_J \colon J \hookrightarrow I \coprod J$ are the inclusions. In particular, when $I = \{1, \dots, k\}$, $J = \{1, \dots, k'\}$, we have

$$\boxtimes \colon \mathcal{S}\mathrm{et}^k_\Delta \times \mathcal{S}\mathrm{et}^{k'}_\Delta \to \mathcal{S}\mathrm{et}^{k+k'}_\Delta$$

by the formula $S \boxtimes S' = (\mathbf{\Delta}^\iota)^* S \times (\mathbf{\Delta}^{\iota'})^* S'$, where $\iota \colon \{1, \dots, k\} \hookrightarrow \{1, \dots, k + k'\}$ is the identity and $\iota' \colon \{1, \dots, k'\} \hookrightarrow \{1, \dots, k+k'\}$ sends j to $j+k$. In other words, $(S \boxtimes S')_{n_1\cdots n_{k+k'}} = S_{n_1\cdots n_k} \times S'_{n_{k+1}\cdots n_{k+k'}}$. We have $\Delta^{n_1} \boxtimes \cdots \boxtimes \Delta^{n_k} = \Delta^{n_1,\dots,n_k}$.

Definition 1.3.2 (Multi-marked simplicial set). An *I-marked simplicial set* (resp. *I-marked ∞-category*) is the data $(X, \mathcal{E} = \{\mathcal{E}_i\}_{i\in I})$, where X is a simplicial set (resp. an ∞-category) and, for all $i \in I$, $\mathcal{E}_i$ is a set of edges of X which contains every degenerate edge. A morphism $f \colon (X, \{\mathcal{E}_i\}_{i\in I}) \to (X', \{\mathcal{E}'_i\}_{i\in I})$ of I-marked simplicial sets is a map $f \colon X \to X'$ having the property that $f(\mathcal{E}_i) \subseteq \mathcal{E}'_i$ for all $i \in I$. We denote the category of I-marked simplicial sets by $\mathcal{S}\mathrm{et}^{I+}_\Delta$. It is the strict fiber product of I copies of $\mathcal{S}\mathrm{et}^+_\Delta$ above $\mathcal{S}\mathrm{et}_\Delta$.

Consider the functor $\delta^*_{I+}\colon \mathrm{Set}^I_\Delta \to \mathrm{Set}^{I+}_\Delta$ sending S to $(\delta^*_I S, \{\mathcal{E}_i\}_{i\in I})$, where $\mathcal{E}_i$ is the set of edges of $\epsilon^I_i S \subseteq \delta^*_I S$. This functor admits a right adjoint $\delta^{I+}_*\colon \mathrm{Set}^{I+}_\Delta \to \mathrm{Set}^I_\Delta$ sending $(X, \{\mathcal{E}_i\}_{i\in I})$ to the I-simplicial subset of $\delta^I_* X$ whose $(n_i)_{i\in I}$-cells are maps $\Delta^{[n_i]_{i\in I}} \to X$ such that for every $i \in I$ and every map $\Delta^1 \to \epsilon^I_i \Delta^{n_i|i\in I}$, the composition

$$\Delta^1 \to \epsilon^I_i \Delta^{\{n_i\}_{i\in I}} \to \Delta^{n_i|i\in I} \to X$$

is in $\mathcal{E}_i$. An *k-marked simplicial set* is a $\{1,\dots,k\}$-marked simplicial set. We will write k instead of I in the notations Set^{I+}_Δ [5], δ^*_{I+} and δ^{I+}_*.

Notation 1.3.3.

(1) For an I-marked simplicial set $(X, \mathcal{E} = \{\mathcal{E}_i\}_{i\in I})$, we simply let
$$X_{\mathcal{E}} = X_{\{\mathcal{E}_i\}_{i\in I}} = \delta^{I+}_*(X, \{\mathcal{E}_i\}_{i\in I}).$$
In particular, for any marked simplicial set $(X, \mathcal{E})$, $X_{\mathcal{E}} \subseteq X$ is the simplicial subset spanned by the edges in $\mathcal{E}$.

(2) For an I-marked ∞-category $(\mathcal{C}, \mathcal{E})$, we denote by $\mathcal{C}^{\mathrm{cart}}_{\mathcal{E}}$ the *Cartesian I-simplicial nerve* of $(\mathcal{C}, \mathcal{E})$ ([27, 3.8 (4)]). Briefly speaking, its $(n_i)_{i\in I}$-cells are functors $\Delta^{[n_i]_{i\in I}} \to \mathcal{C}$ such that the image of a morphism in the i-th direction is in $\mathcal{E}_i$ for $i \in I$, and the image of every "unit square" is a Cartesian square. When $\mathcal{E}_i = \mathcal{C}_1$ for every $i \in I$, we simply write $\mathcal{C}^{\mathrm{cart}}_{|I}$ for $\mathcal{C}^{\mathrm{cart}}_{\mathcal{E}}$.

1.4. **Partial adjoints.**

Definition 1.4.1. Consider diagrams of ∞-categories

$$\begin{tikzcd} \mathcal{C} \arrow[d, "U"'] \arrow[dr, phantom, "\sigma"] & \mathcal{D} \arrow[l, "F"'] \arrow[d, "V"] \\ \mathcal{C}' & \mathcal{D}' \arrow[l, "F'"] \end{tikzcd} \qquad \begin{tikzcd} \mathcal{C} \arrow[r, "G"] \arrow[d, "U"'] \arrow[dr, phantom, "\tau"] & \mathcal{D} \arrow[d, "V"] \\ \mathcal{C}' \arrow[r, "G'"] & \mathcal{D}' \end{tikzcd}$$

which commute up to specified equivalences $\alpha\colon F' \circ V \to U \circ F$ and $\beta\colon V \circ G \to G' \circ U$. We say that σ is a *left adjoint* to τ and τ is a *right adjoint* to σ, if F is a left adjoint of G, F' is a left adjoint of G', and α is equivalent to the composed transformation

$$F' \circ V \to F' \circ V \circ G \circ F \xrightarrow{\beta} F' \circ G' \circ U \circ F \to U \circ F.$$

The diagram τ has a left adjoint if and only if τ is left adjointable in the sense of [29, 7.3.1.2] and [30, 6.2.3.13]. If G and G' are equivalences, then τ is left adjointable. We have analogous notions for ordinary categories. A square τ of ∞-categories is left adjointable if and only if G and G' admit left adjoints and the square $\mathrm{h}\tau$ of homotopy categories is left adjointable. When visualizing a square $\Delta^1 \times \Delta^1 \to \mathcal{C}$, we adopt the convention that the first factor of $\Delta^1 \times \Delta^1$ is vertical and the second factor is horizontal.

Lemma 1.4.2. *Consider a diagram of right Quillen functors*

$$\begin{tikzcd} \mathcal{C} \arrow[r, "G"] \arrow[d, "U"'] & \mathcal{D} \arrow[d, "V"] \\ \mathcal{C}' \arrow[r, "G'"] & \mathcal{D}' \end{tikzcd}$$

of model categories, which commutes up to a natural equivalence $\beta\colon V \circ G \to G' \circ U$ and is endowed with Quillen equivalences (F, G) and (F', G'). Assume that U preserves weak equivalences and all objects of $\mathcal{D}'$ are cofibrant. Let α be the composed transformation

$$F' \circ V \to F' \circ V \circ G \circ F \xrightarrow{\beta} F' \circ G' \circ U \circ F \to U \circ F.$$

Then for every fibrant-cofibrant object Y of $\mathcal{D}$, the morphism $\alpha(Y)\colon (F' \circ V)(Y) \to (U \circ F)(Y)$ is a weak equivalence.

[5] In particular, Set^{2+}_Δ in our notation is Set^{++}_Δ in [30, 6.2.3.2].

Proof. The square $R\beta$

$$\begin{tikzcd} \mathrm{h}\mathcal{C} \arrow[r, "RG"] \arrow[d, "RU"'] & \mathrm{h}\mathcal{D} \arrow[d, "RV"] \\ \mathrm{h}\mathcal{C}' \arrow[r, "RG'"] & \mathrm{h}\mathcal{D}' \end{tikzcd}$$

of homotopy categories is left adjointable. Let $\sigma\colon LF' \circ RV \to RU \circ LF$ be its left adjoint. For fibrant-cofibrant Y, $\alpha(Y)$ computes $\sigma(Y)$. $\square$

We apply Lemma 1.4.2 to the straightening functor [29, 3.2.1]. Let $p\colon S' \to S$ be a morphism of simplicial sets and $\pi\colon \mathcal{C}' \to \mathcal{C}$ be a functor of simplicial categories fitting into a diagram

$$\begin{tikzcd} \mathfrak{C}[S'] \arrow[r, "\phi'"] \arrow[d, "\mathfrak{C}[p]"'] & \mathcal{C}'^{op} \arrow[d, "\pi^{op}"] \\ \mathfrak{C}[S] \arrow[r, "\phi"] & \mathcal{C}^{op} \end{tikzcd}$$

which is commutative up to a simplicial natural equivalence. By [29, 3.2.1.4], we have a diagram

$$\begin{tikzcd} (\mathcal{S}\mathrm{et}_\Delta^+)^{\mathcal{C}} \arrow[r, "\mathrm{Un}_\phi^+"] \arrow[d, "\pi^*"'] & (\mathcal{S}\mathrm{et}_\Delta^+)_{/S} \arrow[d, "p^*"] \\ (\mathcal{S}\mathrm{et}_\Delta^+)^{\mathcal{C}'} \arrow[r, "\mathrm{Un}_{\phi'}^+"] & (\mathcal{S}\mathrm{et}_\Delta^+)_{/S'}, \end{tikzcd}$$

which satisfies the assumptions of Lemma 1.4.2 if ϕ and ϕ' are equivalences of simplicial categories. In this case, for every fibrant object $f\colon X \to S$ of $(\mathcal{S}\mathrm{et}_\Delta^+)_{/S}$, endowed with the Cartesian model structure, the morphism

$$(St_{\phi'}^+ \circ p^*)X \to (\pi^* \circ St_\phi^+)X$$

is a pointwise Cartesian equivalence. Similarly, if $g\colon \mathcal{C} \to \mathcal{D}$ is a functor of ($\mathcal{V}$-small) categories, [29, 3.2.5.14] provides a diagram

$$\begin{tikzcd} (\mathcal{S}\mathrm{et}_\Delta^+)^{\mathcal{D}} \arrow[r, "\mathrm{N}_\bullet^+(\mathcal{D})"] \arrow[d, "g^*"'] & (\mathcal{S}\mathrm{et}_\Delta^+)_{/\mathrm{N}(\mathcal{D})} \arrow[d, "\mathrm{N}(g)^*"] \\ (\mathcal{S}\mathrm{et}_\Delta^+)^{\mathcal{C}} \arrow[r, "\mathrm{N}_\bullet^+(\mathcal{C})"] & (\mathcal{S}\mathrm{et}_\Delta^+)_{/\mathrm{N}(\mathcal{C})} \end{tikzcd}$$

satisfying the assumptions of Lemma 1.4.2. Thus for every fibrant object Y of $(\mathcal{S}\mathrm{et}_\Delta^+)_{/\mathrm{N}(\mathcal{D})}$, endowed with the coCartesian model structure, the morphism

$$\mathfrak{F}^+_{\mathrm{N}(g)^*Y}(\mathcal{C}) \to g^*\mathfrak{F}^+_Y(\mathcal{D})$$

is a pointwise coCartesian equivalence.

Proposition 1.4.3. *Consider quadruples (I, J, R, f) where I is a set, $J \subseteq I$, R is an I-simplicial set and $f\colon \delta_I^* R \to \mathcal{C}\mathrm{at}_\infty$ is a functor, satisfying the following conditions:*

(1) *For every $j \in J$ and every edge e of $\epsilon_j^I R$, the functor $f(e)$ has a left adjoint.*
(2) *For all $i \in J^c = I \backslash J$, $j \in J$, $\tau \in (\epsilon_{i,j}^I R)_{11}$, the square $f(\tau)\colon \Delta^1 \times \Delta^1 \to \mathcal{C}\mathrm{at}_\infty$ is left adjointable.*

There exists a way to associate, to every such quadruple, a functor $f_J\colon \delta_{I,J}^* R \to \mathcal{C}\mathrm{at}_\infty$, satisfying the following conclusions:

(1) *$f_J \mid \delta_{J^c}^*(\mathbf{\Delta}^\iota)_* R = f \mid \delta_{J^c}^*(\mathbf{\Delta}^\iota)_* R$, where $\iota\colon J^c \to J$ is the inclusion.*
(2) *For every $j \in J$ and every edge e of $\epsilon_j^I R$, the functor $f_J(e)$ is a left adjoint of $f(e)$.*
(3) *For all $i \in J^c$, $j \in J$, $\tau \in (\epsilon_{i,j}^I R)_{11}$, $f_J(\tau)$ is a left adjoint of $f(\tau)$.*
(4) *For two quadruples (I, J, R, f), (I', J', R', f') and maps $\mu\colon I' \to I$, $u\colon (\mathbf{\Delta}^\mu)^* R' \to R$ such that $J' = \mu^{-1}(J)$ and $f' = f \circ \delta_I^* u$, the functor $f'_{J'}$ is equivalent to $f_J \circ \delta_{I,J}^* u$.*

When visualizing $(1,1)$-cells of $\epsilon^I_{i,j}R$, we adopt the convention that direction i is vertical and direction j is horizontal. If J^c is nonempty, then assumption (2) implies assumption (1), and conclusion (3) implies conclusion (2).

Proof. Recall that we have fixed a fibrant replacement functor $\mathrm{Fibr}\colon \mathcal{S}\mathrm{et}^+_\Delta \to \mathcal{S}\mathrm{et}^+_\Delta$. Let $\sigma \in (\delta^*_{I,J}R)_n$ be an object of $\mathbf{\Delta}_{/\delta^*_{I,J}R}$, corresponding to $\Delta^{n_i|i\in I}_J \to R$, where $n_i = n$. It induces a functor $f(\sigma)\colon \mathrm{N}(D) \simeq \Delta^{[n_i]_{i\in I}}_J \to \mathcal{C}\mathrm{at}_\infty$, where D is the partially ordered set $S \times T^{op}$, $S = [n]^{J^c}$, $T = [n]^J$. This corresponds to a projectively fibrant simplicial functor $\mathcal{F}\colon \mathfrak{C}[\mathrm{N}(D)] \to \mathcal{S}\mathrm{et}^+_\Delta$. Let $\phi_D\colon \mathfrak{C}[\mathrm{N}(D)] \to D$ be the canonical equivalence of simplicial categories and let $\mathcal{F}' = (\mathrm{Fibr}^D \circ St^+_{\phi_D^{op}} \circ \mathrm{Un}^+_{\mathrm{N}(D)^{op}})\mathcal{F}\colon D \to \mathcal{S}\mathrm{et}^+_\Delta$. We have weak equivalences

$$\mathcal{F} \leftarrow (St^+_{\mathrm{N}(D)^{op}} \circ \mathrm{Un}^+_{\mathrm{N}(D)^{op}})\mathcal{F} \\ \to (\phi^*_D \circ \phi_{D!} \circ St^+_{\mathrm{N}(D)^{op}} \circ \mathrm{Un}^+_{\mathrm{N}(D)^{op}})\mathcal{F} \simeq (\phi^*_D \circ St^+_{\phi^{op}_D} \circ \mathrm{Un}^+_{\mathrm{N}(D)^{op}})\mathcal{F} \to \phi^*_D(\mathcal{F}').$$

Thus, for every $\tau \in (\epsilon^I_{i,j}\mathrm{N}(D))_{1,1}$, $\mathcal{F}'(\tau)$ is equivalent to $f(\tau)$. Let $\mathcal{F}''$ be the composition

$$S \to (\mathcal{S}\mathrm{et}^+_\Delta)^{T^{op}} \xrightarrow{\mathrm{Un}^+_{\phi_T}} (\mathcal{S}\mathrm{et}^+_\Delta)_{/\mathrm{N}(T)},$$

where the first functor is induced by $\mathcal{F}'$. For every $s \in S$, $\mathcal{F}''(s)\colon X \to \mathrm{N}(T)$ is fibrant for the Cartesian model structure. In other words, there exists a Cartesian fibration $p\colon Y \to \mathrm{N}(T)$ and an isomorphism $X \simeq Y^\natural$. By assumption (1), for every morphism $t \to t'$ of T, the induced functor $Y_{t'} \to Y_t$ has a left adjoint. By [29, 5.2.2.5], p is also a coCartesian fibration. We consider the object $(p, \mathcal{E})$ of $(\mathcal{S}\mathrm{et}^+_\Delta)_{/\mathrm{N}(T)}$, where $\mathcal{E}$ is the set of p-coCartesian edges. By assumption (2), this construction is functorial in s, giving rise to a functor $\mathcal{G}'\colon S \to (\mathcal{S}\mathrm{et}^+_\Delta)_{/\mathrm{N}(T)}$. The composition

$$S \xrightarrow{\mathcal{G}'} (\mathcal{S}\mathrm{et}^+_\Delta)_{/\mathrm{N}(T)} \xrightarrow{\mathfrak{F}^+_\bullet(T)} (\mathcal{S}\mathrm{et}^+_\Delta)^T \xrightarrow{\mathrm{Fibr}^T} (\mathcal{S}\mathrm{et}^+_\Delta)^T$$

induces a projectively fibrant diagram

$$\mathcal{G}\colon S \times T \to \mathcal{S}\mathrm{et}^+_\Delta.$$

We denote by $\mathcal{G}_\sigma\colon [n] \to \mathcal{S}\mathrm{et}^+_\Delta$ the composition

$$[n] \to S \times T \to \mathcal{S}\mathrm{et}^+_\Delta,$$

where the first functor is the diagonal functor. The construction of $\mathcal{G}_\sigma$ is not functorial in σ because the straightening functors do not commute with pullbacks, even up to natural equivalences. Nevertheless, for every morphism $d\colon \sigma \to \tilde{\sigma}$ in $\mathbf{\Delta}_{/\delta^*_{I,J}R}$, we have a canonical morphism $\mathcal{G}_\sigma \to d^*\mathcal{G}_{\tilde{\sigma}}$ in $(\mathcal{S}\mathrm{et}^+_\Delta)^{[n]}$, which is a weak equivalence by Lemma 1.4.2. The functor

$$(\mathbf{\Delta}_{/\delta^*_{I,J}R})_{\sigma/} \to (\mathcal{S}\mathrm{et}^+_\Delta)^{[n]}$$

sending $d\colon \sigma \to \tilde{\sigma}$ to $d^*\mathcal{G}_{\tilde{\sigma}}$ induces a map

$$\mathcal{N}(\sigma) := \mathrm{N}((\mathbf{\Delta}_{/\delta^*_{I,J}R})_{\sigma/}) \to \mathrm{Map}^\sharp((\Delta^n)^\flat, (\mathcal{C}\mathrm{at}_\infty)^\natural),$$

which we denote by $\Phi(\sigma)$. Since the category $(\mathbf{\Delta}_{/\delta^*_{I,J}R})_{\sigma/}$ has an initial object, $\mathcal{N}(\sigma)$ is weakly contractible. This construction is functorial in σ so that $\Phi\colon \mathcal{N} \to \mathrm{Map}[\delta^*_{I,J}R, \mathcal{C}\mathrm{at}_\infty]$ is a morphism of $(\mathcal{S}\mathrm{et}_\Delta)^{(\mathbf{\Delta}_{/\delta^*_{I,J}R})^{op}}$. Applying Lemma 1.2.1 (1), we obtain a functor $\widetilde{f_J}\colon \delta^*_{I,J}R \to \mathcal{C}\mathrm{at}_\infty$ satisfying (2), (3) up to homotopy.

Under the assumptions of (4), $\delta^*_{I,J}u\colon \delta^*_{I',J'}R' \to \delta^*_{I,J}R$ induces $\varphi\colon \mathcal{N}' \to (\delta^*_{I,J}u)^*\mathcal{N}$. By construction, there exists a homotopy between Φ' and $((\delta^*_{I,J}u)^*\Phi) \circ \varphi$. By Lemma 1.2.1 (2), this implies that $\widetilde{f'_{J'}}$ and $\widetilde{f_J} \circ \delta^*_{I,J}u$ are homotopic.

By construction, there exists a homotopy between $r^*\Phi$ and the composed map $r^*\mathcal{N} \to \Delta^0_Q \xrightarrow{f|Q} \mathrm{Map}[Q, \mathcal{C}\mathrm{at}_\infty]$, where $Q = \delta^*_{J^c}(\mathbf{\Delta}^\iota)_*R$ and $r\colon Q \to \delta^*_{I,J}R$ is the inclusion. By Lemma 1.2.1 (2), this

implies that $\widetilde{f_J} \mid Q$ and $f \mid Q$ are homotopic. Since the inclusion

$$Q^\natural \times (\Delta^1)^\sharp \coprod_{Q^\natural \times (\Delta^{\{0\}})^\sharp} (\delta^*_{I,J} R)^\natural \times (\Delta^{\{0\}})^\sharp \to (\delta^*_{I,J} R)^\natural \times (\Delta^1)^\sharp$$

is marked anodyne, there exists $f_J\colon \delta^*_{I,J} R \to \mathcal{C}\mathrm{at}_\infty$ homotopic to $\widetilde{f_J}$ such that $f_J \mid Q = f \mid Q$. □

Remark 1.4.4.

(1) There is an obvious dual version of Proposition 1.4.3 for right adjoints.
(2) Proposition 1.4.3 holds without the assumption that R is $\mathcal{V}$-small. To see this, it suffices to apply the proposition to the composite map $\delta^*_I R \xrightarrow{f} \mathcal{C}\mathrm{at}_\infty \to \mathcal{C}\mathrm{at}^{\mathcal{W}}_\infty$, where $\mathcal{W} \supseteq \mathcal{V}$ is a universe containing R and $\mathcal{C}\mathrm{at}^{\mathcal{W}}_\infty$ is the ∞-category of ∞-categories in $\mathcal{W}$.
(3) Applying Proposition 1.4.3 (and Remark 1.4.4 (2)) to the quadruple $(2, \{1\}, \delta_{2*}\mathcal{C}\mathrm{at}_\infty, f)$, where $f\colon \delta_2^*\delta_{2*}\mathcal{C}\mathrm{at}_\infty \to \mathcal{C}\mathrm{at}_\infty$ is the counit map, we get a universal morphism $\delta^*_{2,\{1\}}\delta_{2*}\mathcal{C}\mathrm{at}_\infty \to \mathcal{C}\mathrm{at}_\infty$. In fact, for any quadruple (I', J', R', f'), if we denote by $\mu\colon I' \to \{1,2\}$ the map given by $\mu^{-1}(1) = J'$, then $f'\colon \delta_2^*(\mathbf{\Delta}^\mu)^* R' \to \mathcal{C}\mathrm{at}_\infty$ uniquely determines a map $u\colon (\mathbf{\Delta}^\mu)^* R' \to \delta_{2*}\mathcal{C}\mathrm{at}_\infty$ by adjunction and $f'_{J'}$ can be taken to be the composite map
$$\delta^*_{I,J} R' \simeq \delta^*_{2,\{1\}} (\mathbf{\Delta}^\mu)^* R' \xrightarrow{\delta^*_{2,\{1\}} u} \delta^*_{2,\{1\}}\delta_{2*}\mathcal{C}\mathrm{at}_\infty \xrightarrow{f} \mathcal{C}\mathrm{at}_\infty.$$
(4) For the quadruple $(1, \{1\}, \mathcal{P}\mathrm{r}^{\mathrm{R}}, f)$ where $f\colon \mathcal{P}\mathrm{r}^{\mathrm{R}} \to \mathcal{C}\mathrm{at}_\infty$ is the natural inclusion, the map f_J constructed in Proposition 1.4.3 induces an equivalence $f_{\mathcal{P}\mathrm{r}}\colon (\mathcal{P}\mathrm{r}^{\mathrm{R}})^{op} \to \mathcal{P}\mathrm{r}^{\mathrm{L}}$. This gives another proof of the second assertion of [29, 5.5.3.4]. By restriction, this equivalence induces an equivalence $f_{\mathcal{P}\mathrm{r}_{\mathrm{st}}}\colon \mathcal{P}\mathrm{r}^{\mathrm{L}}_{\mathrm{st}} \to (\mathcal{P}\mathrm{r}^{\mathrm{R}}_{\mathrm{st}})^{op}$.
(5) For the quadruple $(2, \{1\}, S^{op} \boxtimes \mathrm{Fun}^{\mathrm{LAd}}(S^{op}, \mathcal{C}\mathrm{at}_\infty), f)$ where
$$f\colon S^{op} \times \mathrm{Fun}^{\mathrm{LAd}}(S^{op}, \mathcal{C}\mathrm{at}_\infty) \to \mathcal{C}\mathrm{at}_\infty$$
is the natural map, then the map $f_J\colon S \times \mathrm{Fun}^{\mathrm{LAd}}(S^{op}, \mathcal{C}\mathrm{at}_\infty) \to \mathcal{C}\mathrm{at}_\infty$ constructed in Proposition 1.4.3 induces an equivalence $\mathrm{Fun}^{\mathrm{LAd}}(S^{op}, \mathcal{C}\mathrm{at}_\infty) \to \mathrm{Fun}^{\mathrm{RAd}}(S, \mathcal{C}\mathrm{at}_\infty)$. This gives another proof of [30, 6.2.3.18 (3)].

1.5. **Symmetric monoidal ∞-categories.** Let $\mathcal{F}\mathrm{in}_*$ be the category of pointed finite sets defined in [30, 2.0.0.2]. It is (equivalent to) the category whose objects are sets $\langle n \rangle = \langle n \rangle^\circ \cup \{*\}$, where $\langle n \rangle^\circ = \{1, \dots, n\}$ ($\langle 0 \rangle = \emptyset$) for $n \geq 0$, and morphisms are maps of sets that map $*$ to $*$.

Let $\mathfrak{P} = (M', T, \{p_\alpha\colon \Lambda^2_0 \to \mathrm{N}(\mathcal{F}\mathrm{in}_*)\}_{\alpha\in A})$ be the categorical pattern on the simplicial set $\mathrm{N}(\mathcal{F}\mathrm{in}_*)$ defined in [30, 2.1.4.13]. Let $\mathfrak{P}_0 = (M', T, \emptyset)$ be the canonical categorical pattern [30, B.0.20]. We endow $(\mathcal{S}\mathrm{et}^+_\Delta)_{/\mathfrak{P}}$ and $(\mathcal{S}\mathrm{et}^+_\Delta)_{/\mathfrak{P}_0}$ with the left proper combinatorial simplicial model structures constructed in [30, B.0.19]. The latter coincides with the coCartesian model structure on $(\mathcal{S}\mathrm{et}^+_\Delta)_{/\mathrm{N}(\mathcal{F}\mathrm{in}_*)}$. Let $\mathcal{C}\mathrm{at}^\otimes_\infty = \mathrm{N}(((\mathcal{S}\mathrm{et}^+_\Delta)_{/\mathfrak{P}})^\circ)$ be the *∞-category of symmetric monoidal ∞-categories* [30, 2.1.4.13]. Since $\mathfrak{P}$-filtered objects [30, B.0.18] are automatically $\mathfrak{P}_0$-filtered, $\mathcal{C}\mathrm{at}^\otimes_\infty \subseteq \mathrm{N}(((\mathcal{S}\mathrm{et}^+_\Delta)_{/\mathfrak{P}_0})^\circ)$ is a full subcategory (spanned by symmetric monoidal ∞-categories). Moreover, the functor
$$\mathrm{N}(\mathrm{N}^+_\bullet(\mathcal{F}\mathrm{in}_*))\colon \mathrm{N}(((\mathcal{S}\mathrm{et}^+_\Delta)^{\mathcal{F}\mathrm{in}_*})^\circ) \to \mathrm{N}(((\mathcal{S}\mathrm{et}^+_\Delta)_{/\mathrm{N}(\mathcal{F}\mathrm{in}_*)})^\circ)$$
is a categorical equivalence by [29, 3.2.5.18, A.3.1.12], and the functor
$$\mathrm{N}(((\mathcal{S}\mathrm{et}^+_\Delta)^{\mathcal{F}\mathrm{in}_*})^\circ) \to \mathrm{Fun}(\mathrm{N}(\mathcal{F}\mathrm{in}_*), \mathrm{N}((\mathcal{S}\mathrm{et}^+_\Delta)^\circ)) \simeq \mathrm{Fun}(\mathrm{N}(\mathcal{F}\mathrm{in}_*), \mathcal{C}\mathrm{at}_\infty)$$
is a categorical equivalence by [29, 4.2.4.4]. Together, they provide a categorical equivalence $\phi\colon \mathrm{N}(((\mathcal{S}\mathrm{et}^+_\Delta)_{/\mathfrak{P}_0})^\circ) \to \mathrm{Fun}(\mathrm{N}(\mathcal{F}\mathrm{in}_*), \mathcal{C}\mathrm{at}_\infty)$.

Notation 1.5.1. For an ∞-category $\mathcal{C}$, we denote by $\mathrm{Mon}_{\mathrm{Comm}}(\mathcal{C}) \subseteq \mathrm{Fun}(\mathrm{N}(\mathcal{F}\mathrm{in}_*), \mathcal{C})$ the full subcategory spanned by the commutative monoid objects of $\mathcal{C}$ [30, 2.4.2.2]. A functor $X\colon \mathrm{N}(\mathcal{F}\mathrm{in}_*) \to \mathcal{C}$ is an object of $\mathrm{Mon}_{\mathrm{Comm}}(\mathcal{C})$ if and only if for each $n \geq 0$, the functors $\{X(\rho^i)\colon X(\langle n \rangle) \to X(\langle 1 \rangle)\}_{1\leq i\leq n}$ exhibits $X(\langle n \rangle)$ as an n-fold product of $X(\langle 1 \rangle)$, where ρ^i is defined in [30, 2.0.0.2].

The functor ϕ restricts to a categorical equivalence $\mathcal{C}\mathrm{at}_\infty^\otimes \to \mathrm{Mon}_{\mathrm{Comm}}(\mathcal{C}\mathrm{at}_\infty)$. In what follows, we will generally not distinguish between $\mathcal{C}\mathrm{at}_\infty^\otimes$ and $\mathrm{Mon}_{\mathrm{Comm}}(\mathcal{C}\mathrm{at}_\infty)$. There is a forgetful functor $\mathrm{G}\colon \mathcal{C}\mathrm{at}_\infty^\otimes \to \mathcal{C}\mathrm{at}_\infty$ assigning to each symmetric monoidal ∞-category $\mathcal{C}^\otimes$ its underlying ∞-category $\mathcal{C} = \mathcal{C}^\otimes(\langle 1\rangle)$. The unique active map $\langle 2\rangle \to \langle 1\rangle$ [30, 2.1.2.1] induces a functor

$$-\otimes -\colon \mathcal{C}\times\mathcal{C} \simeq \mathcal{C}^\otimes(\langle 2\rangle) \to \mathcal{C}^\otimes(\langle 1\rangle) = \mathcal{C}.$$

The ∞-category $\mathcal{C}\mathrm{at}_\infty^\otimes$ admits limits and colimits and such limits and colimits are preserved by G. For two symmetric monoidal ∞-categories $\mathcal{C}^\otimes$ and $\mathcal{D}^\otimes$, the ∞-category of symmetric monoidal functors is denoted by $\mathrm{Fun}^\otimes(\mathcal{C}^\otimes, \mathcal{D}^\otimes)$ [30, 2.1.3.7].

Example 1.5.2. Let $\mathrm{Mon}_{\mathrm{Comm}}$ be the category of commutative monoids. There is an isomorphism $\mathrm{N}(\mathrm{Mon}_{\mathrm{Comm}}) \xrightarrow{\sim} \mathrm{Mon}_{\mathrm{Comm}}(\mathrm{N}(\mathcal{S}\mathrm{et}))$ sending M to the functor $\langle n\rangle \mapsto M^n$. The fully faithful inclusion $\mathrm{N}(\mathcal{S}\mathrm{et}) \subseteq \mathcal{C}\mathrm{at}_\infty$ induces a fully faithful functor $\mathrm{N}(\mathrm{Mon}_{\mathrm{Comm}}) \to \mathrm{Mon}_{\mathrm{Comm}}(\mathcal{C}\mathrm{at}_\infty)$. For a commutative monoid M, we denote its image in $\mathcal{C}\mathrm{at}_\infty^\otimes$ by $M^\otimes$.

Recall that a symmetric monoidal ∞-category $\mathcal{C}^\otimes$ is *closed* [30, 4.1.1.17] if the functor $-\otimes-\colon \mathcal{C}\times\mathcal{C}\to\mathcal{C}$, written as $\mathcal{C} \to \mathrm{Fun}(\mathcal{C},\mathcal{C})$, factors through $\mathrm{Fun}^{\mathrm{L}}(\mathcal{C},\mathcal{C})$.

Notation 1.5.3. We define a subcategory $\mathcal{P}\mathrm{r}^{\mathrm{L}\otimes}$ (resp. $\mathcal{P}\mathrm{r}^{\mathrm{L}\otimes}_{\mathrm{st}}$) of $\mathcal{C}\mathrm{at}_\infty^\otimes$ as follows:

- An object is a symmetric monoidal ∞-categories $\mathcal{C}^\otimes$ such that $\mathcal{C} = \mathrm{G}(\mathcal{C}^\otimes)$ is presentable (resp. and stable).
- A morphism is a symmetric monoidal functor $F^\otimes\colon \mathcal{C}^\otimes \to \mathcal{D}^\otimes$ such that the underlying functor $F = \mathrm{G}(F^\otimes)$ is a left adjoint functor.

Moreover, we define $\mathcal{P}\mathrm{r}^{\mathrm{L}\otimes}_{\mathrm{cl}} \subseteq \mathcal{P}\mathrm{r}^{\mathrm{L}\otimes}$ (resp. $\mathcal{P}\mathrm{r}^{\mathrm{L}\otimes}_{\mathrm{st,cl}} \subseteq \mathcal{P}\mathrm{r}^{\mathrm{L}\otimes}_{\mathrm{st}}$) to be the full subcategory spanned by closed symmetric monoidal ∞-categories. The functor G restricts to a functor $\mathcal{P}\mathrm{r}^{\mathrm{L}\otimes}_{\mathrm{st,cl}} \to \mathcal{P}\mathrm{r}^{\mathrm{L}}_{\mathrm{st}}$ sending $\mathcal{C}^\otimes$ to its underlying ∞-category $\mathcal{C}$.

Lemma 1.5.4. *The ∞-category $\mathcal{P}\mathrm{r}^{\mathrm{L}\otimes}_{\mathrm{cl}}$ admits small limits and such limits are preserved by the composite functor $\mathcal{P}\mathrm{r}^{\mathrm{L}\otimes}_{\mathrm{cl}} \subseteq \mathcal{C}\mathrm{at}_\infty^\otimes \xrightarrow{\mathrm{G}} \mathcal{C}\mathrm{at}_\infty$. The same holds for $\mathcal{P}\mathrm{r}^{\mathrm{L}\otimes}_{\mathrm{st,cl}}$.*

Proof. By the fact that both $\mathrm{G}\colon \mathcal{C}\mathrm{at}_\infty^\otimes \to \mathcal{C}\mathrm{at}_\infty$ and $\mathcal{P}\mathrm{r}^{\mathrm{L}} \subseteq \mathcal{C}\mathrm{at}_\infty$ preserve small limits, we only need to show that for a small simplicial set S and a diagram $p^\otimes\colon S \to \mathcal{P}\mathrm{r}^{\mathrm{L}\otimes}$ such that $p^\otimes(s) = \mathcal{C}_s^\otimes$ is closed for every vertex s of S, the limit $\varprojlim(p^\otimes)$ is closed.

Let $p\colon S \to \mathcal{P}\mathrm{r}^{\mathrm{L}\otimes} \to \mathcal{P}\mathrm{r}^{\mathrm{L}}$ (resp. $p'\colon S \to \mathcal{P}\mathrm{r}^{\mathrm{L}\otimes} \to \mathrm{Fun}(\Delta^1, \mathcal{C}\mathrm{at}_\infty)$) be the diagram induced by restriction to the object $\langle 1\rangle$ (resp. unique active map $\langle 2\rangle \to \langle 1\rangle$) of $\mathrm{N}(\mathcal{F}\mathrm{in}_*)$. For every object c of $\mathcal{C} = \varprojlim(p)$, it induces a diagram $p'_c\colon S \to \mathrm{Fun}(\Delta^1, \mathcal{P}\mathrm{r}^{\mathrm{L}}_{\mathrm{st}})$ such that $p'_c(s)$ is the functor $f_s^* c \otimes -\colon \mathcal{C}_s \to \mathcal{C}_s$ that admits right adjoints, where $f_s^*\colon \mathcal{C} \to \mathcal{C}_s$ is the obvious functor. Since $\mathcal{P}\mathrm{r}^{\mathrm{L}} \subseteq \mathcal{C}\mathrm{at}_\infty$ is stable under small limits. The limit $\varprojlim(p'_c)$ is an object of $\mathrm{Fun}^{\mathrm{L}}(\mathcal{C},\mathcal{C})$, which confirms the lemma. □

Remark 1.5.5. A diagram $p\colon S^\triangleleft \to \mathcal{C}\mathrm{at}_\infty^\otimes$ is a limit diagram if and only if $\mathrm{G}\circ p\colon S^\triangleleft \to \mathcal{C}\mathrm{at}_\infty^\otimes \xrightarrow{\mathrm{G}} \mathcal{C}\mathrm{at}_\infty$ is a limit diagram, by the dual version of [29, 5.1.2.3].

Our next goal is to introduce an ∞-operad [30, 2.1.1.10] $\mathcal{P}\mathrm{f}^\otimes$, which will be used in the enhanced operation map to encode the projection formula.

Definition 1.5.6. We define a colored operad [30, 2.1.1.1] $\mathbf{Pf}$ whose set of objects is $\{\mathfrak{a}, \mathfrak{m}\}$ as follows:

$$\mathrm{Mul}_{\mathbf{Pf}}(\{X_i\}, Y) = \begin{cases} \{*\} & \text{if } \#\{i \mid X_i = \mathfrak{m}\} = 0,\ Y = \mathfrak{a}, \\ \{*\} & \text{if } \#\{i \mid X_i = \mathfrak{m}\} = 1,\ Y = \mathfrak{m}, \\ \emptyset & \text{otherwise.} \end{cases}$$

We let $\mathbf{Pf}^\otimes$ denote the category obtained by applying [30, 2.1.1.7] to $\mathbf{Pf}$, and $\mathcal{P}\mathrm{f}^\otimes$ denote the ∞-operad $\mathrm{N}(\mathbf{Pf}^\otimes)$ [30, 2.1.1.21].

Remark 1.5.7. Consider the ∞-category $\mathcal{K}_{\mathrm{Comm}} \subseteq \mathrm{Fun}(\Delta^1, \mathrm{N}(\mathcal{F}\mathrm{in}_*))$ [30, 3.3.2.1]. We have $\mathcal{K}_{\mathrm{Comm}} \times_{\mathrm{Fun}(\{0\},\mathrm{N}(\mathcal{F}\mathrm{in}_*))} \mathrm{Fun}(\{0\},\{\langle 1\rangle\}) \simeq \mathrm{N}(\mathcal{F}\mathrm{in}_*)_{\langle 1\rangle/}$. The functor $(\mathcal{F}\mathrm{in}_*)_{\langle 1\rangle/} \to \mathbf{Pf}^{\otimes}$ sending $\alpha\colon \langle 1\rangle \to \langle n\rangle$ to $(\langle n\rangle, (X_i)_{1\le i\le m})$, where

$$X_i = \begin{cases} \mathfrak{m} & \text{if } i \in \operatorname{Im}\alpha, \\ \mathfrak{a} & \text{otherwise,} \end{cases}$$

identifies $(\mathcal{F}\mathrm{in}_*)_{\langle 1\rangle/}$ with a full subcategory of $\mathbf{Pf}^{\otimes}$. The induced functor $\mathrm{N}(\mathcal{F}\mathrm{in}_*)_{\langle 1\rangle/} \to \mathcal{P}\mathrm{f}^{\otimes}$ is an approximation to $\mathcal{P}\mathrm{f}^{\otimes}$ [30, 2.3.3.6]. Let $\mathcal{O}^{\otimes}$ be an ∞-operad. We let $\mathrm{Mod}(\mathcal{O})$ denote the underlying ∞-category of $\mathrm{Mod}(\mathcal{O})^{\otimes} = \mathrm{Mod}^{\mathrm{Comm}}(\mathcal{O})^{\otimes}$ [30, 3.3.3.8, 4.4.1.1]. Unwinding the definitions and applying [30, 2.3.3.23 (1)], we obtain an equivalence of ∞-categories $\mathrm{Alg}_{\mathcal{P}\mathrm{f}}(\mathcal{O}) \to \mathrm{Mod}(\mathcal{O})$, where $\mathrm{Alg}_{\mathcal{P}\mathrm{f}}(\mathcal{O})$ is the category of ∞-operad maps from $\mathcal{P}\mathrm{f}^{\otimes}$ to $\mathcal{O}^{\otimes}$ [30, 2.1.2.7].

Notation 1.5.8. We introduce a series of notations.

(1) Let $\mathcal{C}$ be a category. We extend $\mathcal{C}$ to a colored operad, still denoted by $\mathcal{C}$, by the formula $\mathrm{Mul}_{\mathcal{C}}(\{X_i\}_{1\le i\le m}, Y) = \prod_{i=1}^m \mathrm{Hom}_{\mathcal{C}}(X,Y)$. We let $\mathcal{C}^{\amalg}$ denote the category obtained by applying [30, 2.1.1.7] to this colored operad. An object of $\mathcal{C}^{\amalg}$ is a pair $(\langle m\rangle, (X_i)_{1\le i\le m})$, where $\langle m\rangle$ is an object of $\mathcal{F}\mathrm{in}_*$, X_i is an object of $\mathcal{C}$. A morphism $(\langle m\rangle, (X_i)_{1\le i\le m}) \to (\langle n\rangle, (X'_j)_{1\le j\le n})$ of $\mathcal{C}^{\amalg}$ is a pair $(\alpha, (f_i)_{i\in\alpha^{-1}(\langle n\rangle^\circ)})$, where $\alpha\colon \langle m\rangle \to \langle n\rangle$ is a morphism of $\mathcal{F}\mathrm{in}_*$, $f_i\colon X_i \to X'_{\alpha(i)}$ is a morphism of $\mathcal{C}$. By definition, $\mathrm{N}(\mathcal{C}^{\amalg})$ is isomorphic to the simplicial set $\mathrm{N}(\mathcal{C})^{\amalg}$ defined in [30, 2.4.3.1].

(2) For $\mathcal{C} = [1]$, we represent $(X_i)_{1\le i\le m}$ by the set $S \subseteq \langle m\rangle^\circ$ of indices i for which $X_i = 1$. Under this convention, an object of $[1]^{\amalg}$ is a pair $(\langle m\rangle, S)$, where $\langle m\rangle$ is an object of $\mathcal{F}\mathrm{in}_*$, $S \subseteq \langle m\rangle^\circ$. A morphism $(\langle m\rangle, S) \to (\langle n\rangle, T)$ is a morphism $\alpha\colon \langle m\rangle \to \langle n\rangle$ of $\mathcal{F}\mathrm{in}_*$ such that $\alpha(S) \subseteq T \cup \{*\}$.

The colored operad map $\mathbf{Pf} \to [1]$ sending $\mathfrak{a}$ to 0 and $\mathfrak{m}$ to 1 induces a functor $\mathbf{Pf}^{\otimes} \to [1]^{\amalg}$, which allows us to identify $\mathbf{Pf}^{\otimes}$ with the subcategory of $[1]^{\amalg}$ whose objects are the same as the objects of $[1]^{\amalg}$, and whose morphisms are the morphisms $\alpha\colon (\langle m\rangle, S) \to (\langle n\rangle, T)$ in $[1]^{\amalg}$ such that α induces a bijection $S \cap \alpha^{-1}(T) \to T$. Under this identification, the functor $\mathcal{P}\mathrm{f}^{\otimes} \to \mathrm{N}(\mathcal{F}\mathrm{in}_*)$ is the forgetful functor. The induced injection $\mathcal{P}\mathrm{f}^{\otimes} \to (\Delta^1)^{\amalg}$ is an ∞-operad map [30, 2.1.2.7].

(3) By [30, 2.4.2.5, 2.4.3.18], the map

$$\Delta^1 \times \mathrm{N}(\mathcal{F}\mathrm{in}_*) \to (\Delta^1)^{\amalg} \qquad (0, \langle n\rangle) \mapsto (\langle n\rangle, \emptyset) \quad (1, \langle n\rangle) \mapsto (\langle n\rangle, \langle n\rangle^\circ)$$

induces an equivalence of ∞-categories

$$\mathrm{Mon}_{(\Delta^1)^{\amalg}}(\mathcal{C}\mathrm{at}_\infty) \to \mathrm{Fun}(\Delta^1, \mathrm{Mon}_{\mathrm{Comm}}(\mathcal{C}\mathrm{at}_\infty)).$$

Taking a quasi-inverse and restricting to $\mathcal{P}\mathrm{f}^{\otimes}$, we obtain a functor

$$\mathrm{pf}\colon \mathrm{Fun}((\Delta^1)^{op}, \mathcal{C}\mathrm{at}_\infty^{\otimes}) \simeq \mathrm{Fun}(\Delta^1, \mathcal{C}\mathrm{at}_\infty^{\otimes}) \to \mathrm{Mon}_{\mathcal{P}\mathrm{f}}(\mathcal{C}\mathrm{at}_\infty),$$

where $\mathrm{Mon}_{\mathcal{P}\mathrm{f}}(\mathcal{C}\mathrm{at}_\infty) \subseteq \mathrm{Fun}(\mathcal{P}\mathrm{f}^{\otimes}, \mathcal{C}\mathrm{at}_\infty)$ is the full subcategory of $\mathcal{P}\mathrm{f}^{\otimes}$-monoids in $\mathcal{C}\mathrm{at}_\infty$ [30, 2.4.2.1].

We define a subcategory $\mathrm{Mon}_{\mathcal{P}\mathrm{f}}^{\mathcal{P}\mathrm{r}^{\mathrm{L}}_{\mathrm{st}}}(\mathcal{C}\mathrm{at}_\infty) \subseteq \mathrm{Mon}_{\mathcal{P}\mathrm{f}}(\mathcal{C}\mathrm{at}_\infty)$ as follows:

- The objects of $\mathrm{Mon}_{\mathcal{P}\mathrm{f}}^{\mathcal{P}\mathrm{r}^{\mathrm{L}}_{\mathrm{st}}}(\mathcal{C}\mathrm{at}_\infty)$ are monoids $M\colon \mathcal{P}\mathrm{f}^{\otimes} \to \mathcal{C}\mathrm{at}_\infty$ such that $M(X)$ is a *presentable stable ∞-category* for every object X of $\mathcal{P}\mathrm{f}^{\otimes}$.
- A morphism $F\colon M \to N$ of $\mathcal{P}\mathrm{f}^{\otimes}$-monoids in $\mathcal{C}\mathrm{at}_\infty$ is in $\mathrm{Mon}_{\mathcal{P}\mathrm{f}}^{\mathcal{P}\mathrm{r}^{\mathrm{L}}_{\mathrm{st}}}(\mathcal{C}\mathrm{at}_\infty)$ if and only if $F(X)\colon M(X) \to N(X)$ admits right adjoints for every object X of $\mathcal{P}\mathrm{f}^{\otimes}$.

This subcategory is stable under small limits. Moreover, pf induces a functor

$$\mathrm{pf}\colon \mathrm{Fun}((\Delta^1)^{op}, \mathcal{P}\mathrm{r}^{\mathrm{L}\otimes}_{\mathrm{st}}) \simeq \mathrm{Fun}(\Delta^1, \mathcal{P}\mathrm{r}^{\mathrm{L}\otimes}_{\mathrm{st}}) \to \mathrm{Mon}_{\mathcal{P}\mathrm{f}}^{\mathcal{P}\mathrm{r}^{\mathrm{L}}_{\mathrm{st}}}(\mathcal{C}\mathrm{at}_\infty). \tag{1.1}$$

(4) For any object X of $\mathcal{P}\mathrm{f}^{\otimes}$, we denote by $\mathrm{G}_X\colon \mathrm{Mon}_{\mathcal{P}\mathrm{f}}^{\mathcal{P}\mathrm{r}^{\mathrm{L}}_{\mathrm{st}}}(\mathcal{C}\mathrm{at}_\infty) \to \mathcal{P}\mathrm{r}^{\mathrm{L}}_{\mathrm{st}}$ the functor given by evaluation at X. Similarly, for any morphism α in $\mathcal{P}\mathrm{f}^{\otimes}$, we denote by $\mathrm{G}_\alpha\colon \mathrm{Mon}_{\mathcal{P}\mathrm{f}}^{\mathcal{P}\mathrm{r}^{\mathrm{L}}_{\mathrm{st}}}(\mathcal{C}\mathrm{at}_\infty) \to \mathrm{Fun}(\Delta^1, \mathcal{C}\mathrm{at}_\infty)$ the functor given by evaluation at α. We will often apply this to the map $\zeta\colon (\langle 2\rangle, \{1\}) \to (\langle 1\rangle, \{1\})$ sending both 1 and 2 to 1, which is a morphism in $\mathcal{P}\mathrm{f}^{\otimes}$.

Remark 1.5.9. A diagram $p\colon K^{\triangleleft} \to \mathrm{Mon}_{\mathcal{P}\mathrm{f}^{\otimes}}^{\mathcal{P}\mathrm{r}^{\mathrm{L}}_{\mathrm{st}}}(\mathcal{C}\mathrm{at}_{\infty})$ is a limit diagram if and only if $\mathrm{G}_{(\langle 1\rangle,\emptyset)} \circ p$ and $\mathrm{G}_{(\langle 1\rangle,\{1\})} \circ p$ are limit diagrams.

The ∞-operad map $\mathrm{N}(\mathcal{F}\mathrm{in}_*) \to \mathcal{P}\mathrm{f}^{\otimes}$ sending $\langle n\rangle$ to $(\langle n\rangle,\emptyset)$ induces a functor $\mathrm{Mon}_{\mathcal{P}\mathrm{f}}^{\mathcal{P}\mathrm{r}^{\mathrm{L}}_{\mathrm{st}}}(\mathcal{C}\mathrm{at}_{\infty}) \to \mathcal{P}\mathrm{r}^{\mathrm{L}\otimes}_{\mathrm{st}}$. By construction, the composite map

$$\mathrm{Fun}((\Delta^1)^{op}, \mathcal{P}\mathrm{r}^{\mathrm{L}\otimes}_{\mathrm{st}}) \xrightarrow{\mathrm{pf}} \mathrm{Mon}_{\mathcal{P}\mathrm{f}}^{\mathcal{P}\mathrm{r}^{\mathrm{L}}_{\mathrm{st}}}(\mathcal{C}\mathrm{at}_{\infty}) \to \mathcal{P}\mathrm{r}^{\mathrm{L}\otimes}_{\mathrm{st}}$$

is equivalent to $\mathrm{Fun}((d^1_0)^{op}, \mathcal{P}\mathrm{r}^{\mathrm{L}\otimes}_{\mathrm{st}})$, so that the composite map

$$\mathcal{P}\mathrm{r}^{\mathrm{L}\otimes}_{\mathrm{st}} \xrightarrow{\mathrm{diag}} \mathrm{Fun}((\Delta^1)^{op}, \mathcal{P}\mathrm{r}^{\mathrm{L}\otimes}_{\mathrm{st}}) \xrightarrow{\mathrm{pf}} \mathrm{Mon}_{\mathcal{P}\mathrm{f}}^{\mathcal{P}\mathrm{r}^{\mathrm{L}}_{\mathrm{st}}}(\mathcal{C}\mathrm{at}_{\infty}) \to \mathcal{P}\mathrm{r}^{\mathrm{L}\otimes}_{\mathrm{st}}$$

is equivalent to the identity.

All the above discussions remain valid if we replace $\mathcal{P}\mathrm{r}^{\mathrm{L}\otimes}_{\mathrm{st}}$ by $\mathcal{P}\mathrm{r}^{\mathrm{L}\otimes}_{\mathrm{st,cl}}$, which will actually be the case we use below, and we will keep the same notations.

2. Enhanced operations for ringed topoi

In this chapter, we construct a map $\mathbf{T}^{\otimes}$ (2.1) that enhances the derived $*$-pullback and derived tensor product for ringed topoi. It also encodes the symmetric monoidal structures in a homotopy-coherent way. This serves as a starting point for the construction of the enhanced operation map.

The construction is based on the flat model structure. This marks a major difference with the study of quasi-coherent sheaves. For the latter one can simply start with the dual version of the model structure constructed in [30, 1.3.4.3], because the category of quasi-coherent sheaves on affine schemes have enough projectives. The flat model structure for a ringed topological space has been constructed by [14, 15]. In §2.1, we adapt the construction to every topos with enough points.

2.1. The flat model structure.

Let $(X, \mathcal{O}_X)$ be a ringed topos. In other words, X is a (Grothendieck) topos and $\mathcal{O}_X$ is a sheaf of rings in X. An $\mathcal{O}_X$-module C is called *cotorsion* if $\mathrm{Ext}^1(F, C) = 0$ for every flat $\mathcal{O}_X$-module F. The following definition is a special case of [15, 2.1].

Definition 2.1.1. Let K be a cochain complex of $\mathcal{O}_X$-modules.

- K is called a *flat complex* if it is exact and Z^nK is flat for all n.
- K is called a *cotorsion complex* if it is exact and Z^nK is cotorsion for all n.
- K is called a *dg-flat complex* if K^n is flat for every n, and every cochain map $K \to C$, where C is a cotorsion complex, is homotopic to zero.
- K is called a *dg-cotorsion complex* if K^n is cotorsion for every n, and every cochain map $F \to K$, where F is a flat complex, is homotopic to zero.

Lemma 2.1.2. *Let $(f, \gamma)\colon (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ be a morphism of ringed topos. Then*

- *$(f, \gamma)^*$ preserves flat modules, flat complexes, and dg-flat complexes;*
- *$(f, \gamma)_*$ preserves cotorsion modules, cotorsion complexes, and dg-cotorsion complexes.*

Proof. Let $F \in \mathrm{Mod}(Y, \mathcal{O}_Y)$ be flat and $C \in \mathrm{Mod}(X, \mathcal{O}_X)$ be cotorsion. We have a monomorphism $\mathrm{Ext}^1(F, (f, \gamma)_*C) \to \mathrm{Ext}^1((f, \gamma)^*F, C) = 0$. Thus $(f, \gamma)_*C$ is cotorsion. Moreover, since short exact sequences of cotorsion $\mathcal{O}_X$-modules are exact as sequences of presheaves, $(f, \gamma)_*$ preserves short exact sequences of cotorsion modules, hence it preserves cotorsion complexes. It follows that $(f, \gamma)^*$ preserves dg-flat complexes.

It is well known that $(f, \gamma)^*$ preserves flat modules and short exact sequences of flat modules. It follows that $(f, \gamma)^*$ preserves flat complexes and hence $(f, \gamma)_*$ preserves dg-cotorsion complexes □

The model structure in the following generalization of [15, 7.8] is called the *flat model structure*.

Proposition 2.1.3. *Assume that X has enough points. Then there exists a combinatorial model structure on* $\mathrm{Ch}(\mathrm{Mod}(X, \mathcal{O}_X))$ *such that*

- *The cofibrations are the monomorphisms with dg-flat cokernels.*
- *The fibrations are the epimorphisms with dg-cotorsion kernels.*
- *The weak equivalences are quasi-isomorphisms.*

Furthermore, this model structure is monoidal with respect to the usual tensor product of chain complexes.

Remark 2.1.4.
(1) $\mathrm{id}\colon \mathrm{Ch}(\mathrm{Mod}(X,\mathcal{O}_X))^{\mathrm{flat}} \to \mathrm{Ch}(\mathrm{Mod}(X,\mathcal{O}_X))^{\mathrm{inj}}$ is a right Quillen equivalence. Here $\mathrm{Ch}(\mathrm{Mod}(X,\mathcal{O}_X))^{\mathrm{flat}}$ (resp. $\mathrm{Ch}(\mathrm{Mod}(X,\mathcal{O}_X))^{\mathrm{inj}}$) is $\mathrm{Ch}(\mathrm{Mod}(X,\mathcal{O}_X))$ endowed with the flat model structure (resp. the model structure in [30, 1.3.5.3]).
(2) If $X=*$ and $\mathcal{O}_X=R$ is a (commutative) ring, then $\mathrm{id}\colon \mathrm{Ch}(\mathrm{Mod}(*,R))^{\mathrm{proj}} \to \mathrm{Ch}(\mathrm{Mod}(*,R))^{\mathrm{flat}}$ is a *symmetric monoidal* left Quillen equivalence between symmetric monoidal model categories. Here $\mathrm{Ch}(\mathrm{Mod}(*,R))^{\mathrm{proj}}$ is $\mathrm{Ch}(\mathrm{Mod}(*,R))$ endowed with the projective model structure [30, 8.1.2.11].

To prove Proposition 2.1.3, we adapt the proof of [15, 7.8]. Let S be a site, G be a small topologically generating family [2, II 3.0.1] of S. For a presheaf F on S, we put $|F|_G=\sup_{U\in G}\mathrm{card}(F(U))$.

Lemma 2.1.5. *Let $\beta\ge\mathrm{card}(G)$ be an infinite cardinal such that $\beta\ge\mathrm{card}(\mathrm{Hom}(U,V))$ for all U and V in G and κ be a cardinal $\ge 2^\beta$. Let F be a presheaf on S such that $|F|_G\le\kappa$ and let F^+ be the sheaf associated to F. Then $|F^+|_G\le\kappa$.*

Proof. By construction [2, II 3.5], $F^+=LLF$, where $(LF)(U)=\varinjlim_{R\in J(U)}\mathrm{Hom}_{\hat{S}}(R,F)$, $J(U)$ is the set of sieves covering U, $U\in S$, $\hat{S}$ is the category of presheaves on S. By [2, II 3.0.4] and its proof, $|LF|_G\le\beta^2\kappa^{\beta^2}=\kappa$. □

Let $\mathcal{O}_S$ be a sheaf of rings on S. For an element $U\in S$, we denote by $j_{U!}$ the left adjoint of the restriction functor $\mathrm{Mod}(S,\mathcal{O}_S)\to\mathrm{Mod}(U,\mathcal{O}_U)$. Using the fact that $(j_{U!}\mathcal{O}_U)_{U\in G}$ is a family of flat generators of $\mathrm{Mod}(S,\mathcal{O}_S)$, we have the following analogue of [15, 7.7] with essentially the same proof.

Lemma 2.1.6. *Let $\beta\ge\mathrm{card}(G)$ be an infinite cardinal such that $\beta\ge\mathrm{card}(\mathrm{Hom}(U,V))$ for all U and V in G. Let $\kappa\ge\max\{2^\beta,|\mathcal{O}_S|_G\}$ be a cardinal such that $j_{U!}\mathcal{O}_U$ is κ-generated for every U in G. Then the following conditions are equivalent for an $\mathcal{O}_S$-module F.*

- *$|F|_G\le\kappa$.*
- *F is κ-generated.*
- *F is κ-presentable.*

Let F be an $\mathcal{O}_S$-premodule. We say that an $\mathcal{O}_S$-subpremodule $E\subseteq F$ is *G-pure* if $E(U)\subseteq F(U)$ is pure for every U in G. This implies that $E^+\subseteq F^+$ is pure. As in [10, 2.4], one proves the following.

Lemma 2.1.7. *Let $\beta\ge\mathrm{card}(G)$ be an infinite cardinal such that $\beta\ge\mathrm{card}(\mathrm{Hom}(U,V))$ for all U and V in G. Let $\kappa\ge\max\{2^\beta,|\mathcal{O}_S|_G\}$ be a cardinal, $E\subseteq F$ be $\mathcal{O}_S$-premodules such that $|E|_G\le\kappa$. Then there exists a G-pure $\mathcal{O}_S$-subpremodule E' of F containing E such that $|E'|_G\le\kappa$.*

To prove Proposition 2.1.3, we choose a site S of X, and a small topologically generating family G, and a cardinal κ satisfying the assumptions of Lemma 2.1.6. Using the previous lemmas, one shows as in the proof of [15, 7.8] that the conditions of [15, 4.12, 5.1] are satisfied for κ, which finishes the proof.

Remark 2.1.8. Using the sheaves $i_*(\mathbb{Q}/\mathbb{Z})$, where i runs through points $P\to X$ of X, one can show as in [14, 5.6] that a complex K of $\mathcal{O}_X$-modules is dg-flat if and only if K^n is flat for each n and $K\otimes_{\mathcal{O}_X}L$ is exact for each exact sequence L of $\mathcal{O}_X$-modules.

2.2. **Enhanced operations.** Let us start by introducing some notations.

Notation 2.2.1. For $(2,1)$-categories $\mathcal{C}$, $\mathcal{D}$, we denote by $\mathrm{Fun}^{(2,1)}(\mathcal{C},\mathcal{D})$ the $(2,1)$-category of pseudofunctors from $\mathcal{C}$ to $\mathcal{D}$. Morphisms in $\mathrm{Fun}^{(2,1)}(\mathcal{C},\mathcal{D})$ are pseudonatural transformations between pseudofunctors and 2-cells in $\mathrm{Fun}^{(2,1)}(\mathcal{C},\mathcal{D})$ are modifications between pseudonatural transformations. We adopt the convention that pseudofunctors (and pseudonatural transformations) are strictly unital, so that $\mathrm{N}(\mathrm{Fun}^{(2,1)}(\mathcal{C},\mathcal{D}))$ is canonically isomorphic to $\mathrm{Fun}(\mathrm{N}(\mathcal{C}),\mathrm{N}(\mathcal{D}))$.

Example 2.2.2. We will simply write $\mathcal{D}^{\to}$ for $\mathrm{Fun}^{(2,1)}([1],\mathcal{D})$. An object of $\mathcal{D}^{\to}$ is thus a morphism $y\to x$ of $\mathcal{D}$. A morphism of $\mathcal{D}^{\to}$ from $f'\colon y'\to x'$ to $f\colon y\to x$ is a quintuple (u,v,w,α,β), where $u\colon x'\to x$,

$v\colon y'\to y$ and $w\colon y'\to x$ are morphisms in $\mathcal{D}$ and $\alpha\colon w\to f\circ v$, $\beta\colon w\to u\circ f'$ are 2-cells of $\mathcal{D}$, as shown in the following diagram

$$\begin{tikzcd} y' \arrow[r, "f'"] \arrow[d, "v"'] \arrow[dr, "w"] & x' \arrow[d, "u"] \\ y \arrow[r, "f"] & x. \end{tikzcd}$$

A 2-cell $(u_1,v_1,w_1,\alpha_1,\beta_1)\to(u_2,v_2,w_2,\alpha_2,\beta_2)$ of $\mathcal{D}^{\to}$ is a triple $(\epsilon\colon u_1\to u_2,\epsilon'\colon v_1\to v_2,\epsilon''\colon w_1\to w_2)$ of 2-cells of $\mathcal{D}$, compatible with α_i and β_i, $i=1,2$.

Notation 2.2.3. Let $\mathcal{C}$ be an ∞-category, $\mathcal{E}$ be a set of edges of $\mathcal{C}$ which contains all equivalences. We define $\mathcal{E}^{\to}$ to be the set of edges $(y'\xrightarrow{f'}x')\to(y\xrightarrow{f}x)$ of $\mathrm{Fun}(\Delta^1,\mathcal{C})$ corresponding to squares

$$\begin{tikzcd} y' \arrow[r, "f'"] \arrow[d, "q"'] & x' \arrow[d, "p"] \\ y \arrow[r, "f"] & x \end{tikzcd}$$

in $\mathcal{C}$ where p,q are in $\mathcal{E}$. We define $\mathcal{E}^0\subseteq\mathcal{E}^{\to}$ (resp. $\mathcal{E}^1\subseteq\mathcal{E}^{\to}$) to be the set of edges corresponding to the previous squares where p (resp. q) is an equivalence.

Notation 2.2.4. Let $\mathcal{R}\mathrm{inged}\mathcal{P}\mathcal{T}\mathrm{opos}$ be the $(2,1)$-category of ringed $\mathcal{U}$-topoi in $\mathcal{V}$ with enough points:

- An object of E is a ringed topos (X,Λ) such that X has enough points.
- A morphism $(X,\Lambda)\to(X',\Lambda')$ in E is a morphism of ringed topoi in the sense of [2, IV 13.3], namely a pair (f,γ), where $f\colon X\to X'$ is a morphism of topoi and $\gamma\colon f^*\Lambda'\to\Lambda$.
- A 2-morphism $(f_1,\gamma_1)\to(f_2,\gamma_2)$ in E is an equivalence $\epsilon\colon f_1\to f_2$ such that γ_2 equals the composition $f_2^*\Lambda'\xrightarrow{\epsilon^*}f_1^*\Lambda'\xrightarrow{\gamma_1}\Lambda$.
- Composition of morphisms and 2-morphisms are defined in the obvious way.

The functor $(f,\gamma)_*\colon\mathrm{Mod}(X,\Lambda)\to\mathrm{Mod}(X',\Lambda')$ admits a left adjoint $(f,\gamma)^*=\Lambda\otimes_{f^*\Lambda'}f^*-$.

Our goal in this section is to construct a functor

$$\mathbf{T}^{\otimes}\colon \mathrm{N}(\mathcal{R}\mathrm{inged}\mathcal{P}\mathcal{T}\mathrm{opos}^{op})\to\mathcal{P}\mathrm{r}^{\mathrm{L}\otimes}_{\mathrm{st,cl}}, \tag{2.1}$$

where $\mathcal{P}\mathrm{r}^{\mathrm{L}\otimes}_{\mathrm{st,cl}}$ is defined in Notation 1.5.3. It sends

- every object (X,Λ) of $\mathcal{R}\mathrm{inged}\mathcal{P}\mathcal{T}\mathrm{opos}$ to its *derived ∞-category* $\mathcal{D}(X,\Lambda)^{\otimes}$, whose underlying ∞-category $\mathcal{D}(X,\Lambda)$ is the fibrant replacement of $(\mathrm{N}(\mathrm{Ch}(\mathrm{Mod}(X,\Lambda))_{\text{dg-flat}}),W)$. Here $\mathrm{Ch}(\mathrm{Mod}(X,\Lambda))_{\text{dg-flat}}\subseteq\mathrm{Ch}(\mathrm{Mod}(X,\Lambda))$ is the full subcategory spanned by the dg-flat complexes, and W is the set of quasi-isomorphisms.
- every morphism $(f,\lambda)\colon(X,\Lambda)\to(X',\Lambda')$ of $\mathcal{R}\mathrm{inged}\mathcal{P}\mathcal{T}\mathrm{opos}$ to the enhanced pullback functor $(f,\gamma)^*\colon\mathcal{D}(X',\Lambda')^{\otimes}\to\mathcal{D}(X,\Lambda)^{\otimes}$, which is a symmetric monoidal functor.

Let $\mathcal{C}\mathrm{at}_1^+$ be the $(2,1)$-category of *marked categories*, namely pairs $(\mathcal{C},\mathcal{E})$ consisting of an (ordinary) category $\mathcal{C}$ and a set of arrows $\mathcal{E}$ containing all identity arrows. We have a simplicial functor $\mathcal{C}\mathrm{at}_1^+\to\mathcal{S}\mathrm{et}_\Delta^+$ sending $(\mathcal{C},\mathcal{E})$ to $(\mathrm{N}(\mathcal{C}),\mathcal{E})$. We define a 2-functor

$$T^{\otimes}\colon\mathcal{R}\mathrm{inged}\mathcal{P}\mathcal{T}\mathrm{opos}^{op}\to\mathrm{Fun}^{(2,1)}(\mathcal{F}\mathrm{in}_*,\mathcal{C}\mathrm{at}_1^+)$$

as follows. For every object (X,Λ) of $\mathcal{R}\mathrm{inged}\mathcal{P}\mathcal{T}\mathrm{opos}$, $T((X,\Lambda))\colon\mathcal{F}\mathrm{in}_*\to\mathcal{C}\mathrm{at}_1^+$ is the pseudofunctor sending $\langle n\rangle$ to $(\mathrm{Ch}(\mathrm{Mod}(X,\Lambda))_{\text{dg-flat}},W)^n$ and $\alpha\colon\langle m\rangle\to\langle n\rangle$ to the functor

$$\mathrm{Ch}(\mathrm{Mod}(X,\Lambda))_{\text{dg-flat}},W)^m\to\mathrm{Ch}(\mathrm{Mod}(X,\Lambda))_{\text{dg-flat}},W)^n\qquad (K_i)_{1\le i\le m}\mapsto\left(\bigotimes_{\alpha(i)=j}K_i\right)_{1\le j\le n}.$$

For every morphism $(f,\lambda)\colon(X,\Lambda)\to(X',\Lambda')$ of $\mathcal{R}\mathrm{inged}\mathcal{P}\mathcal{T}\mathrm{opos}$, $T((f,\lambda))\colon T((X',\Lambda'))\to T((X,\Lambda))$ is the pseudonatural transformation given by

$$T((f,\lambda))(\langle n\rangle)=((f,\lambda)^*)^n\colon\mathrm{Ch}(\mathrm{Mod}(X',\Lambda'))_{\text{dg-flat}},W')^n\to\mathrm{Ch}(\mathrm{Mod}(X,\Lambda))_{\text{dg-flat}},W)^n.$$

Composing with the simplicial functor $\mathcal{C}\mathrm{at}_1^+ \to \mathcal{S}\mathrm{et}_\Delta^+ \xrightarrow{\mathrm{Fibr}} (\mathcal{S}\mathrm{et}_\Delta^+)^\circ$ and taking nerves, we obtain a map

$$\mathrm{N}(\mathcal{R}\mathrm{inged}\mathcal{P}\mathcal{T}\mathrm{opos}^{op}) \to \mathrm{Fun}(\mathrm{N}(\mathcal{F}\mathrm{in}_*), \mathrm{N}((\mathcal{S}\mathrm{et}_\Delta^+)^\circ)) \simeq \mathrm{Fun}(\mathrm{N}(\mathcal{F}\mathrm{in}_*), \mathcal{C}\mathrm{at}_\infty).$$

By construction, The image is contained in the full subcategory $\mathrm{Mon}_{\mathrm{Comm}}(\mathcal{C}\mathrm{at}_\infty) \simeq \mathcal{C}\mathrm{at}_\infty^\otimes$ (Notation 1.5.1). By construction, the image of (X,Λ), $\mathcal{D}^\otimes(X,\Lambda)$, is a underlying symmetric monoidal ∞-category of $\mathrm{Ch}(\mathrm{Mod}(X,\Lambda))^{\mathrm{flat}}$ [30, 4.1.3.6] and its underlying ∞-category $\mathcal{D}(X,\Lambda)$ is the fibrant replacement of $(\mathrm{N}(\mathrm{Ch}(\mathrm{Mod}(X,\Lambda))_{\text{dg-flat}}), W)$. Therefore, by Remark 2.1.4 (1) and [30, 1.3.4.16, 1.3.5.15], $\mathcal{D}(X,\Lambda)$ is equivalent to the derived ∞-category of $\mathrm{Mod}(X,\Lambda)$ defined in [30, 1.3.5.8], which is a presentable stable ∞-category by [30, 1.3.5.9, 1.3.5.21 (1)]. Combining this with Lemma 1.1.3, we deduce that the image is actually contained in $\mathcal{P}\mathrm{r}^{\mathrm{L}\otimes}_{\mathrm{st,cl}}$. This finishes the construction of (2.1).

By Remark 2.1.4 (2) and [30, 4.1.3.5], for every ring Λ, $\mathcal{D}^\otimes(*,\Lambda)$ is equivalent to the symmetric monoidal ∞-category defined in [30, 8.1.2.12].

Lemma 2.2.5. *The map $\mathbf{T}^\otimes$ (2.1) sends small coproducts to products.*

Proof. This follows from our construction, Remarks 2.1.4, 1.5.5 and [30, 1.3.3.8, 1.3.4.8, 1.3.4.14]. □

Remark 2.2.6. Let $(f,\gamma)\colon (X,\Lambda) \to (X',\Lambda')$ be a morphism in $\mathcal{R}\mathrm{inged}\mathcal{P}\mathcal{T}\mathrm{opos}$. It follows from 2.1.8 and [23, 14.4.1, 18.6.4] that the functors $(f,\gamma)^*\colon \mathrm{D}(X',\Lambda') \to \mathrm{D}(X,\Lambda)$ and $-\otimes-\colon \mathrm{D}(X,\Lambda)\times\mathrm{D}(X,\Lambda) \to \mathrm{D}(X,\Lambda)$ induced by $\mathbf{T}^\otimes$ are equivalent to the respective functors constructed in [23, 18.6], where $\mathrm{D}(X,\Lambda) = \mathrm{h}\mathcal{D}(X,\Lambda)$ and $\mathrm{D}(X',\Lambda') = \mathrm{h}\mathcal{D}(X',\Lambda')$.

3. Enhanced operations for schemes

In this chapter, we construct the enhanced operation map for the category of disjoint union of quasi-compact and separated schemes, and establish several properties of the map. In §3.1, we introduce an abstract notion of (universal) descent and collect some basic properties. In §3.2, we construct the enhanced operation map (3.5) based on the techniques developed in the last chapter. In §3.3, we establish some properties of the map constructed in the previous sections, including an enhanced version of (co)homological descent for smooth coverings. This property is crucial for the extension of the enhanced operation map to algebraic spaces and stacks in Chapter 5.

3.1. **Abstract descent properties.**

Definition 3.1.1 (F-descent). Let $\mathcal{C}$ be an ∞-category admitting pullbacks, $F\colon \mathcal{C}^{op} \to \mathcal{D}$ be a functor of ∞-categories, $f\colon X_0^+ \to X_{-1}^+$ be a morphism of $\mathcal{C}$. We say that f is *of F-descent* if $F\circ(X_\bullet^+)^{op}\colon \mathrm{N}(\mathbf{\Delta}_+) \to \mathcal{D}$ is a limit diagram in $\mathcal{D}$, where $X_\bullet^+\colon \mathrm{N}(\mathbf{\Delta}_+^{op}) \to \mathcal{C}$ is a Čech nerve of f (see the definition after [29, 6.1.2.11]). We say that f is *of universal F-descent* if every pullback of f in $\mathcal{C}$ is of F-descent. Dually, for a functor $G\colon \mathcal{C} \to \mathcal{D}$, we say that f is *of G-codescent* (resp. *of universal G-codescent*) if it is of G^{op}-descent (resp. of universal G^{op}-descent).

We say that a morphism f of an ∞-category $\mathcal{C}$ is a *retraction* if it is a retraction in the homotopy category $\mathrm{h}\mathcal{C}$. Equivalently, f is a retraction if it can be completed into *a weak retraction diagram* [29, 4.4.5.4] $\mathrm{Ret} \to \mathcal{C}$ of $\mathcal{C}$, corresponding to a 2-simplex of $\mathcal{C}$ of the form

$$\begin{tikzcd} & Y \arrow[dr, "f"] & \\ X \arrow[ur, "s"] \arrow[rr, "\mathrm{id}_X"] & & X. \end{tikzcd}$$

The following is an ∞-categorical version of [16, 10.10, 10.11] (for ordinary descent) and [2, Vbis 3.3.1] (for cohomological descent).

Lemma 3.1.2. *Let $\mathcal{C}$ be an ∞-category admitting pullbacks, $F\colon \mathcal{C}^{op} \to \mathcal{D}$ be a functor of ∞-categories. Then*

(1) *Every retraction f in $\mathcal{C}$ is of universal F-descent.*

(2) *Let*

$$\begin{tikzcd} W \arrow[r] \arrow[d, "q"'] & Z \arrow[d, "p"] \\ Y \arrow[r, "f"] & X \end{tikzcd} \tag{3.1}$$

be a pullback diagram in $\mathcal{C}$ such that f and q are of universal F-descent. Then p is of universal F-descent.

(3) *Let*

$$\begin{tikzcd} & Y \arrow[dr, "f"] & \\ Z \arrow[ur, "g"] \arrow[rr, "h"] & & X \end{tikzcd}$$

be a 2-cell of $\mathcal{C}$ such that h is of universal F-descent. Then f is of universal F-descent.

(4) *Let*

$$\begin{tikzcd} & Y \arrow[dr, "f"] & \\ Z \arrow[ur, "g"] \arrow[rr, "h"] & & X \end{tikzcd}$$

be a 2-cell of $\mathcal{C}$ such that f and g are of universal F-descent. Then h is of universal F-descent.

Proof. (1) It suffices to show that f is of F-descent. Consider the map $\mathrm{N}(\mathbf{\Delta}_+^{op})\times\mathrm{Ret}\to\mathcal{C}$, right Kan extension along the inclusion $K=\{[0]\}\times\mathrm{Ret}\coprod_{\{[0]\}\times\{\emptyset\}}\mathrm{N}(\mathbf{\Delta}_+^{\le 0})^{op}\times\{\emptyset\}\subseteq\mathrm{N}(\mathbf{\Delta}_+^{op})\times\mathrm{Ret}$ of the map $K\to\mathcal{C}$ corresponding to the diagram

$$\begin{tikzcd} Y \arrow[rr, "\mathrm{id}_Y"] \arrow[dd, "f"'] & & Y \arrow[dd, "f"] \\ & Y \arrow[dr, "f"] & \\ X \arrow[ur, "s"] \arrow[rr, "\mathrm{id}_X"] & & X. \end{tikzcd}$$

Then by [30, 6.2.1.7], the Čech nerve of f is split. Therefore, the assertion follows from the dual version of [29, 6.1.3.16].

(2) It suffices to show that p is of F-descent. Let $X_{\bullet\bullet}^+\colon\mathrm{N}(\mathbf{\Delta}_+^{op})\times\mathrm{N}(\mathbf{\Delta}_+^{op})\to\mathcal{C}$ be an augmented bisimplicial object of $\mathcal{C}$ such that $X_{\bullet\bullet}^+$ is a right Kan extension of (3.1), considered as a diagram $\mathrm{N}(\mathbf{\Delta}_+^{\le 0})^{op}\times\mathrm{N}(\mathbf{\Delta}_+^{\le 0})^{op}\to\mathcal{C}$. By assumption, $F\circ(X_{i\bullet}^+)^{op}$ is a limit diagram in $\mathcal{D}$ for $i\ge -1$ and $F\circ(X_{\bullet j}^+)^{op}$ is a colimit diagram in $\mathcal{D}$ for $j\ge 0$. By the dual version of [29, 5.5.2.3], $F\circ(X_{\bullet -1}^+)$ is a limit diagram in $\mathcal{D}$, which proves (2) since $X_{\bullet -1}^+$ is a Čech nerve of p.

(3) Consider the diagram

$$\begin{tikzcd} Z \arrow[dr] \arrow[drr, "\mathrm{id}_Z"] \arrow[ddr, "g"'] & & \\ & Y\times_X Z \arrow[r, "\mathrm{pr}_Z"'] \arrow[d, "\mathrm{pr}_Y"] & Z \arrow[d, "h"] \\ & Y \arrow[r, "f"] & X \end{tikzcd} \tag{3.2}$$

in $\mathcal{C}$. Since pr_Z is a retraction, it is of universal F-descent by (1). It then suffices to apply (2).

(4) Consider the diagram (3.2). By (3), pr_Y is of universal F-descent. It then suffices to apply (2). □

Next, we prove a descent lemma for general topoi. Let X be a topos that has enough points, with a fixed final object e. Let $u_0 \colon U_0 \to e$ be a covering, which induces a hypercovering $u_\bullet \colon U_\bullet \to e$ by taking the Čech nerve. Let Λ be a sheaf of rings in X, and $\Lambda_n = \Lambda \times U_n$. In particular, we obtain an augmented simplicial ringed topoi $(X_{/U_\bullet}, \Lambda_\bullet)$, where $U_{-1} = e$ and $\Lambda_{-1} = \Lambda$. Suppose that for every $n \geq -1$, we are given a strictly full subcategory $\mathcal{C}_n$ ($\mathcal{C} = \mathcal{C}_{-1}$) of $\mathrm{Mod}(X_{/U_n}, \Lambda_n)$ such that for every morphism $\alpha \colon [m] \to [n]$ of $\mathbf{\Delta}_+$, $u_\alpha^* \colon \mathrm{Mod}(X_{/U_n}, \Lambda_n) \to \mathrm{Mod}(X_{/U_m}, \Lambda_m)$ sends $\mathcal{C}_n$ to $\mathcal{C}_m$. Then, applying the functor $G \circ \mathbf{T}^{\otimes}$ (2.1), we obtain an augmented cosimplicial ∞-category $\mathcal{D}_{\mathcal{C}_\bullet}(X_{/U_\bullet}, \Lambda_\bullet)$.

Lemma 3.1.3. *Assume that for every object $\mathscr{F}$ of $\mathrm{Mod}(X, \Lambda)$ such that $u_{d_0^0}^* \mathscr{F}$ is in $\mathcal{C}_0$, $\mathscr{F}$ is in $\mathcal{C}$. Then the natural map*

$$\mathcal{D}_{\mathcal{C}}(X, \Lambda) \to \varprojlim_{n \in \mathbf{\Delta}} \mathcal{D}_{\mathcal{C}_n}(X_{/U_n}, \Lambda_n)$$

is a categorical equivalence.

Proof. We first consider the case where $\mathcal{C}_n = \mathrm{Mod}(X_{/U_n}, \Lambda_n)$ for $n \geq -1$. We apply [30, 6.2.4.3]. Assumption (1) follows from the fact that $u_{d_0^0}^* \colon \mathcal{D}(X, \Lambda) \to \mathcal{D}(X_{/U_0}, \Lambda_0)$ is a 1-cell in $\mathcal{P}\mathrm{r}^{\mathrm{L}}_{\mathrm{st}}$. Moreover, the functor $u_{d_0^0}^*$ is conservative since u_0 is a covering. Therefore, we only need to check assumption (2) of [30, 6.2.4.3], that is, the left adjointability of the diagram

$$\begin{tikzcd}
\mathcal{D}(X_{/U_m}, \Lambda_m) \arrow[r, "u^*_{d_0^{m+1}}"] \arrow[d, "u^*_\alpha"'] & \mathcal{D}(X_{/U_{m+1}}, \Lambda_{m+1}) \arrow[d, "u^*_{\alpha'}"] \\
\mathcal{D}(X_{/U_n}, \Lambda_n) \arrow[r, "u^*_{d_0^{n+1}}"] & \mathcal{D}(X_{/U_{n+1}}, \Lambda_{n+1})
\end{tikzcd}$$

for every morphism $\alpha \colon [m] \to [n]$ of $\mathbf{\Delta}_+$, where $\alpha' \colon [m+1] \to [n+1]$ is the induced morphism. This is a special case of Lemma 3.1.4 below.

Now the general case follows from Lemma 3.1.5 and the fact that $u_{d_0^0}^*$ is exact. □

Lemma 3.1.4. *For every Cartesian diagram*

$$\begin{tikzcd}
V' \arrow[r, "f'"] \arrow[d, "q"'] & U' \arrow[d, "p"] \\
V \arrow[r, "f"] & U
\end{tikzcd}$$

of X, the following diagram

(3.3)
$$\begin{tikzcd}
\mathcal{D}(X_{/U}, \Lambda \times U) \arrow[r, "f^*"] \arrow[d, "p^*"'] & \mathcal{D}(X_{/V}, \Lambda \times V) \arrow[d, "q^*"] \\
\mathcal{D}(X_{/U'}, \Lambda \times U') \arrow[r, "f'^*"] & \mathcal{D}(X_{/V'}, \Lambda \times V')
\end{tikzcd}$$

is left adjointable.

Proof. The diagram

$$\begin{tikzcd}
\mathrm{Mod}(X_{/U}, \Lambda \times U) \arrow[d, "p^*"'] & \mathrm{Mod}(X_{/V}, \Lambda \times V) \arrow[l, "f_!"'] \arrow[d, "q^*"] \\
\mathrm{Mod}(X_{/U'}, \Lambda \times U') & \mathrm{Mod}(X_{/V'}, \Lambda \times V') \arrow[l, "f'_!"']
\end{tikzcd}$$

commutes up to isomorphism, and the four functors appearing in it are all exact. Therefore, after replacing $\mathrm{Mod}(X_{/-}, \Lambda \times -)$ by $\mathcal{D}(X_{/-}, \Lambda \times -)$ at all vertices, the above square provides a left adjoint of (3.3). □

Lemma 3.1.5. *Let $p\colon K^{\triangleleft} \to \mathcal{C}\mathrm{at}_\infty$ be a limit diagram. Suppose that for each vertex k of $K^{\triangleleft}$, is given a strictly full subcategory $\mathcal{D}_k \subseteq \mathcal{C}_k = p(k)$ such that*

(1) *For every morphism $f\colon k \to k'$, the induced functor $p(f)$ sends $\mathcal{D}_k$ to $\mathcal{D}_{k'}$.*
(2) *An object c of $\mathcal{C}_\infty$ is in $\mathcal{D}_\infty$ if and only if for every vertex k of K, $p(f_k)(c)$ is in $\mathcal{D}_k$, where ∞ denotes the cone point of $K^{\triangleleft}$, $f_k\colon \infty \to k$ is the unique edge.*

Then the induced diagram $q\colon K^{\triangleleft} \to \mathcal{C}\mathrm{at}_\infty$ sending k to $\mathcal{D}_k$ is also a limit diagram.

Proof. Let $\tilde{p}\colon X \to (K^{op})^{\triangleright}$ be a Cartesian fibration classified by p [29, 3.3.2.2]. Let $Y \subseteq X$ be the simplicial subset spanned by vertices in each fiber X_k that are in the essential image of $\mathcal{D}_k$ for all vertices k of $K^{\triangleleft}$. The map $\tilde{q} = \tilde{p} \,|\, Y\colon Y \to (K^{op})^{\triangleright}$ has the property that if $f\colon x \to y$ is $\tilde{p}$-Cartesian and y is in Y, then x is also in Y by assumption (1), and f is $\tilde{q}$-Cartesian by the dual version of [29, 2.4.1.8]. It follows that $\tilde{q}$ is a Cartesian fibration, which is in fact classified by q. By assumption (2) and [29, 3.3.3.2], q is a limit diagram. □

3.2. **Enhanced operation map.** Let $\mathcal{R}\mathrm{ing}$ be the category of (small) rings. To deal with torsion and adic coefficients simultaneously, we introduce the category $\mathcal{R}\mathrm{ind}$ of ringed diagrams as follows.

Definition 3.2.1 (Ringed diagrams). Define the category $\mathcal{R}\mathrm{ind}$:

- An object of $\mathcal{R}\mathrm{ind}$ is a pair (Ξ, Λ), called a *ringed diagram*, where Ξ is a small partially ordered set and $\Lambda\colon \Xi^{op} \to \mathcal{R}\mathrm{ing}$ is a functor. We identify (Ξ, Λ) with the topos of presheaves on Ξ, ringed by Λ. A typical example is $(\mathbb{N}, n \mapsto \mathbb{Z}/\ell^{n+1}\mathbb{Z})$ with transition maps given by projections.
- A morphism of ringed diagrams $(\Xi, \Lambda) \to (\Xi', \Lambda')$ is a pair (Γ, γ) where $\Gamma\colon \Xi \to \Xi'$ is a functor (that is, an order-preserving map) and $\gamma\colon \Gamma^*\Lambda' := \Lambda' \circ \Gamma^{op} \to \Lambda$ is a morphism in $\mathcal{R}\mathrm{ing}^{\Xi^{op}}$.

For an object (Ξ, Λ) of $\mathcal{R}\mathrm{ind}$ and an object ξ of Ξ, we define the *over ringed diagram* $(\Xi, \Lambda)_{/\xi}$ (resp. *under ringed diagram* $(\Xi, \Lambda)_{\xi/}$) whose underlying category is $\Xi_{/\xi}$ (resp. $\Xi_{\xi/}$) and the corresponding functor is $\Lambda_{/\xi} := \Lambda \,|\, \Xi_{/\xi}$ (resp. $\Lambda_{\xi/} := \Lambda \,|\, \Xi_{\xi/}$).

For any topos X and any small partially ordered set Ξ, we denote by X^{Ξ} the topos $\mathrm{Fun}(\Xi^{op}, X)$. If (Ξ, Λ) is a ringed diagram, then Λ defines a sheaf of rings on X^{Ξ}, which we still denote by Λ. We thus obtain a pseudofunctor

$$\mathcal{PT}\mathrm{opos} \times \mathcal{R}\mathrm{ind} \to \mathcal{R}\mathrm{inged}\mathcal{PT}\mathrm{opos} \tag{3.4}$$

carrying $(X, (\Xi, \Lambda))$ to (X^{Ξ}, Λ), where $\mathcal{PT}\mathrm{opos}$ is the $(2,1)$-category of ringed topoi with enough points.

Notation 3.2.2. For a property (P) in the category $\mathcal{R}\mathrm{ing}$, we say a ringed diagram (Γ, Λ) has the property (P) if for every object ξ of Ξ, the ring $\Lambda(\xi)$ has the property (P). We denote by $\mathcal{R}\mathrm{ind}_{\mathrm{tor}}$ the full subcategory of $\mathcal{R}\mathrm{ind}$ consisting of torsion ringed diagrams.

Let $\mathcal{S}\mathrm{ch}^{\mathrm{qc.sep}} \subseteq \mathcal{S}\mathrm{ch}$ be the full subcategory spanned by (small) disjoint union of quasi-compact and separated schemes. For each object X of $\mathcal{S}\mathrm{ch}$ (resp. $\mathcal{S}\mathrm{ch}^{\mathrm{qc.sep}}$), we denote by $\acute{\mathrm{E}}\mathrm{t}(X) \subseteq \mathcal{S}\mathrm{ch}_{/X}$ (resp. $\acute{\mathrm{E}}\mathrm{t}^{\mathrm{qc.sep}}(X) \subseteq \mathcal{S}\mathrm{ch}^{\mathrm{qc.sep}}_{/X}$) the full subcategory spanned by the étale morphisms. We denote by $X_{\acute{e}t}$ (resp. $X_{\mathrm{qc.sep.\acute{e}t}}$) the associated topos, namely the category of sheaves on $\acute{\mathrm{E}}\mathrm{t}(X)$ (resp. $\acute{\mathrm{E}}\mathrm{t}^{\mathrm{qc.sep}}(X)$). In [2, VII 1.2], $\acute{\mathrm{E}}\mathrm{t}(X)$ is called the étale site of X and $X_{\acute{e}t}$ is called the étale topos of X. The inclusion $\acute{\mathrm{E}}\mathrm{t}^{\mathrm{qc.sep}}(X) \subseteq \acute{\mathrm{E}}\mathrm{t}(X)$ induces an equivalence of topoi $X_{\acute{e}t} \to X_{\mathrm{qc.sep.\acute{e}t}}$. In this chapter, we will often write $X_{\acute{e}t}$ for $X_{\mathrm{qc.sep.\acute{e}t}}$.

Let $A = \mathrm{Ar}(\mathcal{S}\mathrm{ch}^{\mathrm{qc.sep}})$, $F \subseteq A$ be the set of morphisms locally of finite type. The goal of this section is to construct the following *enhanced operation map* (for $\mathcal{S}\mathrm{ch}^{\mathrm{qc.sep}}$):

$$_{\mathcal{S}\mathrm{ch}^{\mathrm{qc.sep}}}\mathrm{EO}\colon \delta^*_{2,\{2\}}\mathrm{Fun}(\Delta^1, \mathrm{N}(\mathcal{S}\mathrm{ch}^{\mathrm{qc.sep}}))^{\mathrm{cart}}_{F^0, A^{\to}} \times \mathrm{N}(\mathcal{R}\mathrm{ind}^{op}_{\mathrm{tor}}) \to \mathrm{Mon}^{\mathcal{P}\mathrm{r}^{\mathrm{L}}_{\mathrm{st}}}_{\mathcal{P}\mathrm{f}}(\mathcal{C}\mathrm{at}_\infty). \tag{3.5}$$

Remark 3.2.3. The map $\mathrm{G}_\zeta \circ {}_{\mathcal{S}\mathrm{ch}^{\mathrm{qc.sep}}}\mathrm{EO}$ (Notation 1.5.8) sends

- a 0-cell $(Y \xrightarrow{f} X, (\Xi, \Lambda))$ to

$$\mathcal{D}(Y^{\Xi}_{\acute{e}t}, \Lambda) \times \mathcal{D}(X^{\Xi}_{\acute{e}t}, \Lambda) \xrightarrow{-\otimes f^*-} \mathcal{D}(Y^{\Xi}_{\acute{e}t}, \Lambda);$$

- a 1-cell

$$\begin{tikzcd} Y' \arrow[r, "f'"] \arrow[d, "q"'] & X', \arrow[d, "p"] & (\Xi, \Lambda) \\ Y \arrow[r, "f"] & X \end{tikzcd}$$

in direction 1 (where p is an isomorphism and q is locally of finite type) to

$$\begin{tikzcd} \mathcal{D}(Y'^{\Xi}_{\text{ét}}, \Lambda) \times \mathcal{D}(X'^{\Xi}_{\text{ét}}, \Lambda) \arrow[r, "-\otimes f'^{*}-"] \arrow[d, "q_! \times p_!"'] & \mathcal{D}(Y'^{\Xi}_{\text{ét}}, \Lambda) \arrow[d, "q_!"] \\ \mathcal{D}(Y^{\Xi}_{\text{ét}}, \Lambda) \times \mathcal{D}(X^{\Xi}_{\text{ét}}, \Lambda) \arrow[r, "-\otimes f^{*}-"] & \mathcal{D}(Y^{\Xi}_{\text{ét}}, \Lambda); \end{tikzcd}$$

- a 1-cell

$$\begin{tikzcd} Y' \arrow[r, "f'"] \arrow[d, "q"'] & X', \arrow[d, "p"] & (\Xi, \Lambda) \\ Y \arrow[r, "f"] & X \end{tikzcd}$$

in direction 2 to

$$\begin{tikzcd} \mathcal{D}(Y'^{\Xi}_{\text{ét}}, \Lambda) \times \mathcal{D}(X'^{\Xi}_{\text{ét}}, \Lambda) \arrow[r, "-\otimes f'^{*}-"] & \mathcal{D}(Y'^{\Xi}_{\text{ét}}, \Lambda) \\ \mathcal{D}(Y^{\Xi}_{\text{ét}}, \Lambda) \times \mathcal{D}(X^{\Xi}_{\text{ét}}, \Lambda) \arrow[r, "-\otimes f^{*}-"] \arrow[u, "q^* \times p^*"] & \mathcal{D}(Y^{\Xi}_{\text{ét}}, \Lambda); \arrow[u, "q^*"'] \end{tikzcd}$$

- a 1-cell $(Y \xrightarrow{f} X, (\Xi', \Lambda') \xrightarrow{(\Gamma,\gamma)} (\Xi, \Lambda))$ in direction 3 to

$$\begin{tikzcd} \mathcal{D}(Y^{\Xi'}_{\text{ét}}, \Lambda') \times \mathcal{D}(X^{\Xi'}_{\text{ét}}, \Lambda') \arrow[r, "-\otimes f^{*}-"] & \mathcal{D}(Y^{\Xi'}_{\text{ét}}, \Lambda') \\ \mathcal{D}(Y^{\Xi}_{\text{ét}}, \Lambda) \times \mathcal{D}(X^{\Xi}_{\text{ét}}, \Lambda) \arrow[r, "-\otimes f^{*}-"] \arrow[u] & \mathcal{D}(Y^{\Xi}_{\text{ét}}, \Lambda) \arrow[u] \end{tikzcd}$$

where the vertical functors are extension of scalars.

Remark 3.2.4. The map ${}_{\mathcal{S}\mathrm{ch}^{\mathrm{qc.sep}}}\mathrm{EO}$ with the target $\mathrm{Mon}^{\mathcal{P}\mathrm{r}^{\mathrm{L}}_{\mathrm{st}}}_{\mathcal{P}\mathrm{f}}(\mathcal{C}\mathrm{at}_\infty)$ encodes more information than the restricted map $\mathrm{G}_\zeta \circ {}_{\mathcal{S}\mathrm{ch}^{\mathrm{qc.sep}}}\mathrm{EO}$. For example, consider a 1-cell

$$\begin{tikzcd} Y \arrow[r, "f"] \arrow[d, "f"'] & X \arrow[d, "\mathrm{id}_X"] \\ X \arrow[r, "\mathrm{id}_X"] & X \end{tikzcd}$$

in direction 1 and fix an object (Ξ, Λ) of $\mathrm{N}(\mathcal{R}\mathrm{ind}^{op}_{\mathrm{tor}})$. Its image under ${}_{\mathcal{S}\mathrm{ch}^{\mathrm{qc.sep}}}\mathrm{EO}$ is a functor $\Delta^1 \times \mathcal{P}\mathrm{f}^{\otimes} \to \mathcal{C}\mathrm{at}_\infty$. By choosing four different decompositions of the active map $(0, (\langle 3\rangle, \{1\})) \to (1, (\langle 1\rangle, \{1\}))$ in the category $[1] \times \mathbf{Pf}$, we obtain a diagram

$$\Delta^1 \times \Delta^1 \to \mathrm{Fun}(\mathcal{D}(Y^{\Xi}_{\text{ét}}, \Lambda) \times \mathcal{D}(X^{\Xi}_{\text{ét}}, \Lambda) \times \mathcal{D}(X^{\Xi}_{\text{ét}}, \Lambda), \mathcal{D}(X^{\Xi}_{\text{ét}}, \Lambda))$$

as

$$\begin{tikzcd} f_!(- \otimes (f^*-) \otimes (f^*-)) \arrow[r, "\sim"] \arrow[d, "\simeq"'] & f_!(- \otimes (f^*-)) \otimes - \arrow[d, "\simeq"] \\ f_!(- \otimes f^*(- \otimes -)) \arrow[r, "\sim"] & (f_!-) \otimes - \otimes -, \end{tikzcd}$$

where all natural transformations are equivalences.

Let $P \subseteq F$ be the set of proper morphisms, $I \subseteq F$ be the set of local isomorphisms.

Lemma 3.2.5. *The map*

$$\delta^*_{3,\{3\}}\mathrm{Fun}(\Delta^1, \mathrm{N}(\mathcal{S}\mathrm{ch}^{\mathrm{qc.sep}}))^{\mathrm{cart}}_{P^0,I^0,A^\to} \to \delta^*_{2,\{2\}}\mathrm{Fun}(\Delta^1, \mathrm{N}(\mathcal{S}\mathrm{ch}^{\mathrm{qc.sep}}))^{\mathrm{cart}}_{F^0,A^\to} \tag{3.6}$$

is a categorical equivalence.

Proof. Let $F_{\mathrm{ft}} \subseteq F$ be the set of morphisms of finite type, $E_{\mathrm{ft}} = E \cap F_{\mathrm{ft}}$, $I_{\mathrm{ft}} = I \cap F_{\mathrm{ft}}$. Consider the following commutative diagram

$$\begin{tikzcd}
\delta^*_{4,\{4\}}\mathrm{Fun}(\Delta^1, \mathrm{N}(\mathcal{S}\mathrm{ch}^{\mathrm{qc.sep}}))^{\mathrm{cart}}_{P^0,I^0_{\mathrm{ft}},I^0,A^\to} \arrow[r] \arrow[d] & \delta^*_{3,\{3\}}\mathrm{Fun}(\Delta^1, \mathrm{N}(\mathcal{S}\mathrm{ch}^{\mathrm{qc.sep}}))^{\mathrm{cart}}_{F^0_{\mathrm{ft}},I^0,A^\to} \arrow[d] \\
\delta^*_{3,\{3\}}\mathrm{Fun}(\Delta^1, \mathrm{N}(\mathcal{S}\mathrm{ch}^{\mathrm{qc.sep}}))^{\mathrm{cart}}_{P^0,I^0,A^\to} \arrow[r] & \delta^*_{2,\{2\}}\mathrm{Fun}(\Delta^1, \mathrm{N}(\mathcal{S}\mathrm{ch}^{\mathrm{qc.sep}}))^{\mathrm{cart}}_{F^0,A^\to}.
\end{tikzcd}$$

To show that the lower horizontal map is a categorical equivalence, it suffices to show that the other three maps are categorical equivalences.

In [27, 6.16], we let

- $\alpha = 1$, $K = \{3,4\}$, $L = \{4\}$;
- $\mathcal{C} = \mathrm{Fun}(\Delta^1, \mathrm{N}(\mathcal{S}\mathrm{ch}^{\mathrm{qc.sep}}))$, $\mathcal{E}_0 = F^0_{\mathrm{ft}}$, $\mathcal{E}_1 = P^0$, $\mathcal{E}_2 = I^0_{\mathrm{ft}}$, $\mathcal{E}_3 = I^0$ and $\mathcal{E}_4 = A^\to$;

Assumption (1) is immediate and assumption (3) is void by [27, 6.17]. By Nagata compactification theorem [7, 4.1], $((\mathrm{Ob}((\mathcal{S}\mathrm{ch}^{\mathrm{qc.sep}})^\to), F^0_{\mathrm{ft}}), P^0, I^0_{\mathrm{ft}})$ satisfies the assumptions of [27, 4.6]. Therefore, assumption (2) (of [27, 6.16]) is also satisfied. It follows that the map in the upper horizontal arrow is a categorical equivalence. Similarly, using [27, 6.16], one proves that the vertical arrows are also categorical equivalences. □

Composing the pseudofunctor $\mathcal{S}\mathrm{ch}^{\mathrm{qc.sep}} \to \mathcal{PT}\mathrm{opos}$ carrying X to $X_{\text{ét}}$ and (3.4), we obtain a pseudofunctor $\mathcal{S}\mathrm{ch}^{\mathrm{qc.sep}} \times \mathcal{R}\mathrm{ind} \to \mathcal{R}\mathrm{inged}\mathcal{PT}\mathrm{opos}$ carrying $(X, (\Xi, \Lambda))$ to $(X^\Xi_{\text{ét}}, \Lambda)$, which induces a functor

$$\mathrm{N}(\mathcal{S}\mathrm{ch}^{\mathrm{qc.sep}}) \times \mathrm{N}(\mathcal{R}\mathrm{ind}) \to \mathrm{N}(\mathcal{R}\mathrm{inged}\mathcal{PT}\mathrm{opos}). \tag{3.7}$$

Composing (3.7) with $\mathbf{T}^\otimes$ (2.1), we get

$$_{\mathcal{S}\mathrm{ch}^{\mathrm{qc.sep}}}\mathrm{EO}^*_\otimes \colon \mathrm{N}(\mathcal{S}\mathrm{ch}^{\mathrm{qc.sep}})^{op} \to \mathrm{Fun}(\mathrm{N}(\mathcal{R}\mathrm{ind}^{op}), \mathcal{P}\mathrm{r}^{\mathrm{L}\otimes}_{\mathrm{st,cl}}). \tag{3.8}$$

Composing with $\mathrm{pf} \circ \mathrm{Fun}((\Delta^1)^{op}, {}_{\mathcal{S}\mathrm{ch}^{\mathrm{qc.sep}}}\mathrm{EO}^*_\otimes)$ (1.1), we obtain a functor

$$_{\mathcal{S}\mathrm{ch}^{\mathrm{qc.sep}}}\mathrm{EO}^*_{\mathrm{pf}} \colon \mathrm{Fun}(\Delta^1, \mathrm{N}(\mathcal{S}\mathrm{ch}^{\mathrm{qc.sep}}))^{op} \times \mathrm{N}(\mathcal{R}\mathrm{ind}^{op}) \to \mathrm{Mon}^{\mathcal{P}\mathrm{r}^{\mathrm{L}}_{\mathrm{st}}}_{\mathcal{P}\mathrm{f}}(\mathcal{C}\mathrm{at}_\infty). \tag{3.9}$$

Consider the composition

$$\begin{aligned}
&\delta^*_4((\mathrm{Fun}(\Delta^1, \mathrm{N}(\mathcal{S}\mathrm{ch}^{\mathrm{qc.sep}}))^{\mathrm{cart}}_{P^0,I^0,A^\to})^{op} \boxtimes \mathrm{N}(\mathcal{R}\mathrm{ind}^{op})) \\
&\qquad \simeq \delta^*_3(\mathrm{Fun}(\Delta^1, \mathrm{N}(\mathcal{S}\mathrm{ch}^{\mathrm{qc.sep}}))^{\mathrm{cart}}_{P^0,I^0,A^\to})^{op} \times \mathrm{N}(\mathcal{R}\mathrm{ind}^{op}) \\
&\qquad\qquad \to \mathrm{Fun}(\Delta^1, \mathrm{N}(\mathcal{S}\mathrm{ch}^{\mathrm{qc.sep}}))^{op} \times \mathrm{N}(\mathcal{R}\mathrm{ind}^{op}) \xrightarrow{_{\mathcal{S}\mathrm{ch}^{\mathrm{qc.sep}}}\mathrm{EO}^*_{\mathrm{pf}}} \mathrm{Mon}^{\mathcal{P}\mathrm{r}^{\mathrm{L}}_{\mathrm{st}}}_{\mathcal{P}\mathrm{f}}(\mathcal{C}\mathrm{at}_\infty),
\end{aligned}$$

which can be written in the form

$$\delta^*_{5,\{1,2,3,4\}}(\mathrm{Fun}(\Delta^1, \mathrm{N}(\mathcal{S}\mathrm{ch}^{\mathrm{qc.sep}}))^{\mathrm{cart}}_{P^0,I^0,A^\to} \boxtimes \mathrm{N}(\mathcal{R}\mathrm{ind}) \boxtimes \mathcal{P}\mathrm{f}^\otimes) \to \mathcal{C}\mathrm{at}_\infty.$$

We apply the dual version of Proposition 1.4.3 for direction 1 to construct

$$\delta^*_{5,\{2,3,4\}}(\mathrm{Fun}(\Delta^1, \mathrm{N}(\mathcal{S}\mathrm{ch}^{\mathrm{qc.sep}}))^{\mathrm{cart}}_{P^0,I^0,A^\to} \boxtimes \mathrm{N}(\mathcal{R}\mathrm{ind}_{\mathrm{tor}}) \boxtimes \mathcal{P}\mathrm{f}^\otimes) \to \mathcal{C}\mathrm{at}_\infty.$$

The adjointability condition, modulo the obvious reduction to the case where Ξ is trivial, is proper base change for directions (1,2) and (1,3), and projection formula for directions (1,4) and (1,5). See [2, XVII 4.3.1] for a proof in D^-. The general case follows by the right completeness of the unbounded derived categories [30, 1.3.4.21] and the fact that $f_*\colon \mathcal{D}(Y,\lambda) \to \mathcal{D}(X,\lambda)$ admits a right adjoint for every

morphism f in P and every object λ of $\mathcal{R}\mathrm{ind}_{\mathrm{tor}}$. We then apply Proposition 1.4.3 for direction 2 to construct

$$\eta\colon \delta^*_{5,\{3,4\}}(\mathrm{Fun}(\Delta^1,\mathrm{N}(\mathcal{S}\mathrm{ch}^{\mathrm{qc.sep}}))^{\mathrm{cart}}_{P^0,I^0,A^{\rightarrow}} \boxtimes \mathrm{N}(\mathcal{R}\mathrm{ind}_{\mathrm{tor}}) \boxtimes \mathcal{P}\mathrm{f}^{\otimes}) \to \mathcal{C}\mathrm{at}_\infty. \tag{3.10}$$

The adjointability condition for direction (2,1) follows from the fact that, for every separated étale morphism f of finite type between quasi-separated and quasi-compact schemes, the functor $f_!$ constructed in [2, XVII 5.1.8] is a left adjoint of f^* [2, XVII 6.2.11]. The adjointability condition for direction (2,3) follows from étale base change. The adjointability conditions for directions (2,4) and (2,5) follow from a trivial projection formula [23, 18.2.5].

Composing (3.10) with a quasi-inverse of (3.6), we obtain (3.5). By construction, ${}_{\mathcal{S}\mathrm{ch}^{\mathrm{qc.sep}}}^{\mathrm{tor}}\mathrm{EO}^*_{\mathrm{pf}} = {}_{\mathcal{S}\mathrm{ch}^{\mathrm{qc.sep}}}\mathrm{EO}^*_{\mathrm{pf}} \mid \mathrm{Fun}(\Delta^1,\mathrm{N}(\mathcal{S}\mathrm{ch}^{\mathrm{qc.sep}}))^{op}\times \mathrm{N}(\mathcal{R}\mathrm{ind}^{op}_{\mathrm{tor}})$ is equivalent to the restriction of ${}_{\mathcal{S}\mathrm{ch}^{\mathrm{qc.sep}}}\mathrm{EO}$ to direction 2. Up to replacing ${}_{\mathcal{S}\mathrm{ch}^{\mathrm{qc.sep}}}\mathrm{EO}$ by an equivalent map, we may assume that the restriction of ${}_{\mathcal{S}\mathrm{ch}^{\mathrm{qc.sep}}}\mathrm{EO}$ to direction 2 is ${}_{\mathcal{S}\mathrm{ch}^{\mathrm{qc.sep}}}^{\mathrm{tor}}\mathrm{EO}^*_{\mathrm{pf}}$.

Variant 3.2.6. Let $Q \subseteq F$ be the set of locally quasi-finite morphisms. Recall that base change for an integral morphism [2, VIII 5.6] holds for all abelian sheaves. Replacing proper base change by finite base change in the construction of (3.5), we obtain

$${}_{\mathcal{S}\mathrm{ch}^{\mathrm{qc.sep}}}^{\mathrm{lqf}}\mathrm{EO}\colon \delta^*_{2,\{2\}}\mathrm{Fun}(\Delta^1,\mathrm{N}(\mathcal{S}\mathrm{ch}^{\mathrm{qc.sep}}))^{\mathrm{cart}}_{Q^0,A^{\rightarrow}} \times \mathrm{N}(\mathcal{R}\mathrm{ind}^{op}) \to \mathrm{Mon}^{\mathcal{P}\mathrm{r}^{\mathrm{L}}_{\mathrm{st}}}_{\mathcal{P}\mathrm{f}}(\mathcal{C}\mathrm{at}_\infty).$$

When restricted to their common domain of definition, this map and ${}_{\mathcal{S}\mathrm{ch}^{\mathrm{qc.sep}}}\mathrm{EO}$ are equivalent.

3.3. **Poincaré duality and (co)homological descent.** The functor $\mathcal{S}\mathrm{ch}^{\mathrm{qc.sep}} \to (\mathcal{S}\mathrm{ch}^{\mathrm{qc.sep}})^{\rightarrow}$ sending X to $X \to \operatorname{Spec}\mathbb{Z}$ induces a map

$$\sigma\colon \mathrm{N}(\mathcal{S}\mathrm{ch}^{\mathrm{qc.sep}})^{\mathrm{cart}}_{F,A} \to \mathrm{Fun}(\Delta^1,\mathrm{N}(\mathcal{S}\mathrm{ch}^{\mathrm{qc.sep}}))^{\mathrm{cart}}_{F^0,A^{\rightarrow}}.$$

We define the *enhanced Base Change map* for $\mathcal{S}\mathrm{ch}^{\mathrm{qc.sep}}$ to be the following composed map

$$\begin{aligned} {}_{\mathcal{S}\mathrm{ch}^{\mathrm{qc.sep}}}\mathrm{EO}^*_!\colon \delta^*_{2,\{2\}}\mathrm{N}(\mathcal{S}\mathrm{ch}^{\mathrm{qc.sep}})^{\mathrm{cart}}_{F,A} &\xrightarrow{\delta^*_{2,\{2\}}\sigma} \delta^*_{2,\{2\}}\mathrm{Fun}(\Delta^1,\mathrm{N}(\mathcal{S}\mathrm{ch}^{\mathrm{qc.sep}}))^{\mathrm{cart}}_{F^0,A^{\rightarrow}} \\ &\xrightarrow{{}_{\mathcal{S}\mathrm{ch}^{\mathrm{qc.sep}}}\mathrm{EO}} \mathrm{Fun}(\mathrm{N}(\mathcal{R}\mathrm{ind}^{op}_{\mathrm{tor}}),\mathrm{Mon}^{\mathcal{P}\mathrm{r}^{\mathrm{L}}_{\mathrm{st}}}_{\mathcal{P}\mathrm{f}}(\mathcal{C}\mathrm{at}_\infty)) \xrightarrow{\mathrm{Fun}(\mathrm{N}(\mathcal{R}\mathrm{ind}^{op}_{\mathrm{tor}}),\mathrm{G}_{(\langle 1\rangle,\{1\})})} \mathrm{Fun}(\mathrm{N}(\mathcal{R}\mathrm{ind}^{op}_{\mathrm{tor}}),\mathcal{P}\mathrm{r}^{\mathrm{L}}_{\mathrm{st}}). \end{aligned} \tag{3.11}$$

Restricting (3.11) to the first direction, we get

$${}_{\mathcal{S}\mathrm{ch}^{\mathrm{qc.sep}}}\mathrm{EO}_!\colon \mathrm{N}(\mathcal{S}\mathrm{ch}^{\mathrm{qc.sep}})_F \to \mathrm{Fun}(\mathrm{N}(\mathcal{R}\mathrm{ind}^{op}_{\mathrm{tor}}),\mathcal{P}\mathrm{r}^{\mathrm{L}}_{\mathrm{st}}).$$

Composing the categorical equivalence $f_{\mathcal{P}\mathrm{r}_{\mathrm{st}}}$ in Remark 1.4.4 with ${}_{\mathcal{S}\mathrm{ch}^{\mathrm{qc.sep}}}\mathrm{EO}_!$, we obtain

$${}_{\mathcal{S}\mathrm{ch}^{\mathrm{qc.sep}}}\mathrm{EO}^!\colon \mathrm{N}(\mathcal{S}\mathrm{ch}^{\mathrm{qc.sep}})^{op}_F \to \mathrm{Fun}(\mathrm{N}(\mathcal{R}\mathrm{ind}_{\mathrm{tor}}),\mathcal{P}\mathrm{r}^{\mathrm{R}}_{\mathrm{st}}).$$

We fix a nonempty set L *of rational primes.* Recall that a ring R is an L-*torsion* ring if each element is killed by an integer that is a product of primes in L. In particular, an L-torsion ring is a torsion ring. We denote by $\mathcal{R}\mathrm{ind}_{\mathsf{L}\text{-tor}} \subseteq \mathcal{R}\mathrm{ind}_{\mathrm{tor}}$ the full subcategory spanned by L-torsion ringed diagrams. Recall that a scheme X is L-*coprime* if L does not contain any residue characteristic of X. Let $\mathcal{S}\mathrm{ch}^{\mathrm{qc.sep}}_{\mathsf{L}}$ be the full subcategory of $\mathcal{S}\mathrm{ch}^{\mathrm{qc.sep}}$ spanned by L-coprime schemes. We let $A_{\mathsf{L}} = \mathrm{Ar}(\mathcal{S}\mathrm{ch}^{\mathrm{qc.sep}}_{\mathsf{L}})$ and $F_{\mathsf{L}} = F \cap A_{\mathsf{L}}$. Moreover, let $L \subseteq F_{\mathsf{L}}$ be the set of smooth morphisms.

Definition 3.3.1 (Shift and twist). We denote by ${}_{\mathcal{S}\mathrm{ch}^{\mathrm{qc.sep}}}\mathrm{EO}^*_{\mathrm{ST}}$ the composite map

$$\mathfrak{C}[\mathrm{N}(\mathcal{S}\mathrm{ch}^{\mathrm{qc.sep}}) \times \mathrm{N}(\mathcal{R}\mathrm{ind})]^{op} \to (\mathcal{S}\mathrm{et}^+_\Delta)^\circ_{/\mathfrak{P}} \to \mathcal{S}\mathrm{et}_\Delta,$$

where the first map is induced by ${}_{\mathcal{S}\mathrm{ch}^{\mathrm{qc.sep}}}\mathrm{EO}^*_{\otimes}$ and adjunction, and the second map sends $\mathcal{D}^{\otimes}$ to $\mathrm{Map}_{(\mathcal{S}\mathrm{et}^+_\Delta)_{/\mathfrak{P}}}(\mathbb{Z}^{\otimes},\mathcal{D}^{\otimes})$, which can be identified with the maximal Kan complex contained in $\mathrm{Fun}^{\otimes}(\mathbb{Z}^{\otimes},\mathcal{D}^{\otimes})$ (see Example 1.5.2 for the notation $\mathbb{Z}^{\otimes}$).

To simply the notations, let $\mathcal{C} = \mathrm{N}(\mathcal{S}\mathrm{ch}^{\mathrm{qc.sep}})$, $\mathcal{L} = \mathrm{N}(\mathcal{R}\mathrm{ind}^{op})$ and ${}_{\mathcal{C}}\mathrm{EO}^*_{\mathrm{ST}} = {}_{\mathcal{S}\mathrm{ch}^{\mathrm{qc.sep}}}\mathrm{EO}^*_{\mathrm{ST}}$. For each integer i, we are going to construct a section $\mathrm{S}_i\mathrm{T}_0$ of $\mathrm{Un}_{\mathcal{C}\times\mathcal{L}^{op}}({}_{\mathcal{C}}\mathrm{EO}^*_{\mathrm{ST}})$. Since $\mathsf{s} = \operatorname{Spec}\mathbb{Z}$ is the final object of $\mathcal{C}$, the restriction map

$$\mathrm{Map}_{\mathcal{C}\times\mathcal{L}^{op}}(\mathcal{C}\times\mathcal{L}^{op},\mathrm{Un}_{\mathcal{C}\times\mathcal{L}^{op}}({}_{\mathcal{C}}\mathrm{EO}^*_{\mathrm{ST}})) \to \mathrm{Map}_{\{\mathsf{s}\}\times\mathcal{L}^{op}}(\{\mathsf{s}\}\times\mathcal{L}^{op},\mathrm{Un}_{\{\mathsf{s}\}\times\mathcal{L}^{op}}({}_{\mathcal{C}}\mathrm{EO}^*_{\mathrm{ST}} \mid \mathfrak{C}[\{\mathsf{s}\}\times\mathcal{L}^{op}])) \tag{3.12}$$

is a trivial fibration. We only need to define an object in the right-hand side of (3.12), which is a Kan complex. For every object $\langle n\rangle$ of $\mathcal{F}\mathrm{in}_*$ and every object $\lambda=(\Xi,\Lambda)$ of $\mathcal{R}\mathrm{ind}$, we have the following functor

$$\mathbb{Z}^n \to (\mathrm{Ch}((\operatorname{Spec}\mathbb{Z})^{\Xi}_{\text{ét}},\Lambda)_{\text{dg-flat}})^n \quad (k_1,\dots,k_n)\mapsto (\cdots\to 0\to\Lambda\to 0\to\cdots)_{1\le m\le n} \tag{3.13}$$

where in the m-th component (which is a dg-flat complex), the constant sheaf Λ in $\mathrm{Mod}((\operatorname{Spec}\mathbb{Z})^{\Xi}_{\text{ét}},\Lambda)$ is put in the degree $-ik_m$. This assignment defines a pseudofunctor from $\mathcal{R}\mathrm{ind}^{op}\times\mathcal{F}\mathrm{in}_*\times[1]$ to $\mathcal{C}\mathrm{at}_1^+$. Taking nerves and applying the unstraightening functor, we obtain an object of $\mathrm{Map}_{\{\mathsf{s}\}\times\mathcal{L}^{op}}(\{\mathsf{s}\}\times\mathcal{L}^{op},\mathrm{Un}_{\{\mathsf{s}\}\times\mathcal{L}^{op}}({}_{\mathcal{C}}\mathrm{EO}^*_{\mathrm{ST}}\mid\mathfrak{C}[\{\mathsf{s}\}\times\mathcal{L}^{op}]))$. Finally, let $\mathrm{S}_i\mathrm{T}_0$ be a lifting of this object to $\mathrm{Fun}_{\mathcal{C}\times\mathcal{L}^{op}}(\mathrm{Un}_{\mathcal{C}\times\mathcal{L}^{op}}({}_{\mathcal{C}}\mathrm{EO}^*_{\mathrm{ST}}))$ via (3.12). If we denote by $\lambda_X\lfloor n\rfloor$ the evaluation of $\mathrm{S}_i\mathrm{T}_0$ at $n\in\mathbb{Z}$ in the fiber above (X,λ), which can be viewed an object of $\mathcal{D}(X,\lambda)=\mathcal{D}(X^{\Xi}_{\text{ét}},\Lambda)$, then the functor $-\otimes\lambda_X\lfloor n\rfloor$ is just (equivalent to) the usual shift by in.

Let i, j be two integers. We let $\mathcal{C}''=\mathcal{S}\mathrm{ch}^{\text{qc.sep}}_{\mathsf{L}}$ and $\mathcal{L}''=\mathrm{N}(\mathcal{R}\mathrm{ind}^{op}_{\mathsf{L}\text{-tor}})$, and repeat the same process for ${}_{\mathcal{C}''}\mathrm{EO}^*_{\mathrm{ST}}:={}_{\mathcal{C}}\mathrm{EO}^*_{\mathrm{ST}}\mid\mathfrak{C}[\mathcal{C}''\times\mathcal{L}''^{op}]$ by taking the final object to be $\operatorname{Spec}\mathbb{Z}[\mathsf{L}^{-1}]$ and modifying (3.13) to be

$$\mathbb{Z}^n\to(\mathrm{Ch}((\operatorname{Spec}\mathbb{Z}[\mathsf{L}^{-1}])^{\Xi}_{\text{ét}},\Lambda)_{\text{dg-flat}})^n \quad (k_1,\dots,k_n)\mapsto(\cdots\to 0\to\Lambda(jk_m)\to 0\to\cdots)_{1\le m\le n},$$

where $\Lambda(jk_m)$ is the jk_m-th Tate twist of Λ in $\mathrm{Mod}((\operatorname{Spec}\mathbb{Z}[\mathsf{L}^{-1}])^{\Xi}_{\text{ét}},\Lambda)$, put in the degree $-ik_m$. The induced section of $\mathrm{Un}_{\mathcal{C}''\times\mathcal{L}''^{op}}({}_{\mathcal{C}''}\mathrm{EO}^*_{\mathrm{ST}})$ is denoted by $\mathrm{S}_i\mathrm{T}_j$. There is only a slight abuse of notation, since when $j=0$ the two separately defined $\mathrm{S}_i\mathrm{T}_0$ are equivalent on their common domain.

For every object $\lambda=(\Xi,\Lambda)$ of $\mathcal{R}\mathrm{ind}$ and every object X of $\mathcal{S}\mathrm{ch}^{\text{qc.sep}}$, we will write $\mathcal{D}(X,\lambda)$ instead of $\mathcal{D}(X^{\Xi}_{\text{ét}},\Lambda)$. There is a t-structure $(\mathcal{D}^{\le 0}(X,\lambda),\mathcal{D}^{\ge 0}(X,\lambda))$[6] on $\mathcal{D}(X,\lambda)$, which induces the usual t-structure on its homotopy category $\mathrm{D}(X^{\Xi}_{\text{ét}},\Lambda)$. We denote by $\tau^{\le 0}$ and $\tau^{\ge 0}$ the corresponding truncation functors. The heart $\mathcal{D}^{\heartsuit}(X,\lambda)\subseteq\mathcal{D}(X,\lambda)$ is canonically equivalent to (the nerve of) the abelian category $\mathrm{Mod}(X^{\Xi}_{\text{ét}},\Lambda)$. The constant sheaf λ_X on X^{Ξ} of value Λ is an object of $\mathcal{D}^{\heartsuit}(X,\lambda)$. Assume that (X,λ) is an object of $\mathrm{N}(\mathcal{S}\mathrm{ch}^{\text{qc.sep}}_{\mathsf{L}})\times\mathrm{N}(\mathcal{R}\mathrm{ind}^{op}_{\mathsf{L}\text{-tor}})$. For every integer d, we denote by $\lambda_X\lfloor d\rfloor$ the evaluation of $\mathrm{S}_2\mathrm{T}_1$ at $d\in\mathbb{Z}$ in the fiber above (X,λ), and let $-\langle d\rangle=-\otimes\lambda_X\lfloor d\rfloor$.

We adapt the classical theory of trace maps and Poincaré duality to the ∞-categorical setting, as follows. Let $f\colon Y\to X$ be a flat morphism, locally of finite presentation, and such that every geometric fiber has dimension $\le d$. Let λ be an object of $\mathrm{N}(\mathcal{R}\mathrm{ind}^{op}_{\mathsf{L}\text{-tor}})$. In [2, XVIII 2.9], Deligne constructed the trace map

$$\mathrm{Tr}_f=\mathrm{Tr}_{f,\lambda}\colon\tau^{\ge 0}f_!\lambda_Y\langle d\rangle\to\lambda_X,$$

which is a 1-cell in $\mathcal{D}^{\heartsuit}(X,\lambda)$.

Remark 3.3.2 (Functoriality of the trace map). The trace maps Tr_f for all such f and λ are functorial in the following sense:

(1) For every morphism $\lambda'\to\lambda$ of $\mathrm{N}(\mathcal{R}\mathrm{ind}^{op}_{\mathsf{L}\text{-tor}})$, the diagram

$$\begin{tikzcd} & \tau^{\ge 0}f_!\lambda_Y\langle d\rangle \arrow[dr, "\mathrm{Tr}_{f,\lambda}"] & \\ \tau^{\ge 0}((\tau^{\ge 0}f_!\lambda'_Y\langle d\rangle)\otimes_{\lambda'_X})\lambda_X \arrow[ur, "\sim"] \arrow[rr, "\tau^{\ge 0}(\mathrm{Tr}_{f,\lambda'}\otimes_{\lambda'_X}\lambda_X)"] & & \lambda_X \end{tikzcd}$$

commutes.

(2) For every Cartesian diagram

$$\begin{tikzcd} Y' \arrow[r, "f'"] \arrow[d, "v"'] & X' \arrow[d, "u"] \\ Y \arrow[r, "f"] & X \end{tikzcd}$$

[6] The reader should be careful that we use a *cohomological* indexing convention, which is different from [30, 1.2.1.4].

in $\mathrm{N}(\mathcal{S}\mathrm{ch}^{\mathrm{qc.sep}}_{\mathsf{L}})$, the diagram

$$\begin{tikzcd} u^*\tau^{\geq 0}f_!\lambda_Y\langle d\rangle \arrow[r, "u^*\mathrm{Tr}_f"] \arrow[d, "\simeq"'] & u^*\lambda_X \arrow[d, "\simeq"] \\ \tau^{\geq 0}f'_!\lambda_{Y'}\langle d\rangle \arrow[r, "\mathrm{Tr}_{f'}"] & \lambda_{X'} \end{tikzcd}$$

commutes.

(3) Consider a 2-cell

$$\begin{tikzcd} Z \arrow[rr, "h"] \arrow[dr, "g"'] & & X \\ & Y \arrow[ur, "f"'] & \end{tikzcd}$$

of $\mathrm{N}(\mathcal{S}\mathrm{ch}^{\mathrm{qc.sep}}_{\mathsf{L}})$ with f (resp. g) flat, locally of finite presentation, and such that every geometric fiber has dimension $\leq d$ (resp. $\leq e$). Then h is flat, locally of finite presentation, and such that every geometric fiber has dimension $\leq d+e$, and the diagram

$$\begin{tikzcd} \tau^{\geq 0}f_!(\tau^{\geq 0}g_!\lambda_Z\langle e\rangle)\langle d\rangle \arrow[r, "\tau^{\geq 0}f_!\mathrm{Tr}_g\langle d\rangle"] \arrow[d, "\simeq"'] & \tau^{\geq 0}f_!\lambda_Y\langle d\rangle \arrow[d, "\mathrm{Tr}_f"] \\ \tau^{\geq 0}h_!\lambda_Z\langle d+e\rangle \arrow[r, "\mathrm{Tr}_h"] & \lambda_X \end{tikzcd}$$

commutes.

Remark 3.3.3. The map ${}_{\mathcal{S}\mathrm{ch}^{\mathrm{qc.sep}}}\mathrm{EO}$ applied to the morphism

$$\begin{tikzcd} Y \arrow[r, "f"] \arrow[d, "f"'] & X \arrow[d, "\mathrm{id}_X"] \\ X \arrow[r, "\mathrm{id}_X"] & X \end{tikzcd}$$

provides the following 2-cell

$$\begin{tikzcd} & \mathcal{D}(Y,\lambda) \arrow[d, "f_!"] \\ \mathcal{D}(X,\lambda) \arrow[ur, "f^*"] \arrow[r, "f_!\lambda_Y\otimes-"'] & \mathcal{D}(X,\lambda). \end{tikzcd}$$

If we abuse of notation by writing $f^*\langle d\rangle$ for $-\langle d\rangle\circ f^*$, then the composition

$$u_f\colon f_!\circ f^*\langle d\rangle \to f_!\lambda_Y\langle d\rangle\otimes - \to \lambda_X\otimes - \to \mathrm{id}_X$$

is a natural transformation, where id_X is the identity functor of $\mathcal{D}(X,\lambda)$, and the second map is induced by the composite map $f_!\lambda_Y\langle d\rangle \to \tau^{\geq 0}f_!\lambda_Y\langle d\rangle \xrightarrow{\mathrm{Tr}_f} \lambda_X$. By [2, XVIII 3.2.5], u_f is a counit transformation when f is smooth and of pure relative dimension d. Therefore, in this case, the functors $f^*\langle d\rangle$ and $f^!$ are equivalent.

Remark 3.3.4. Assume that $f\colon Y\to X$ is flat, locally quasi-finite, and locally of finite presentation. Let λ be an object of $\mathrm{N}(\mathcal{R}\mathrm{ind}^{op})$ (see Variant 3.2.6 for the definition of the enhanced operation map in this setting). In [2, XVII 6.2.3], Deligne constructed the trace map

$$\mathrm{Tr}_f\colon \tau^{\geq 0}f_!\lambda_Y \to \lambda_X,$$

which is a 1-cell in $\mathcal{D}^{\heartsuit}(X,\lambda)$. It coincides with the trace map in Remark 3.3.2 when both are defined, and satisfies similar functorial properties. Moreover, by [2, XVII 6.2.11], the map $u_f\colon f_!\circ f^*\to \mathrm{id}_X$

constructed similarly to Remark 3.3.3 is a counit transform when f is étale. Thus, the functors $f^!$ and f^* are equivalent in this case.

The following proposition will be used in the construction of the enhanced operation map for quasi-separated schemes.

Proposition 3.3.5 ((Co)homological descent). *Let $f\colon X_0^+ \to X_{-1}^+$ be a smooth and surjective morphism in $\mathcal{S}\mathrm{ch}^{\mathrm{qc.sep}}$. Then*

1. *f is of universal ${}_{\mathcal{S}\mathrm{ch}^{\mathrm{qc.sep}}}\mathrm{EO}^*_\otimes$-descent (Definition 3.1.1).*
2. *f is of universal ${}_{\mathcal{S}\mathrm{ch}^{\mathrm{qc.sep}}}\mathrm{EO}_!$-codescent.*

Proof. By [29, 5.1.2.3] and its dual version, we can restrict ${}_{\mathcal{S}\mathrm{ch}^{\mathrm{qc.sep}}}\mathrm{EO}^*_\otimes$ (resp. ${}_{\mathcal{S}\mathrm{ch}^{\mathrm{qc.sep}}}\mathrm{EO}_!$) to a fixed object (Ξ, Λ) of $\mathcal{R}\mathrm{ind}$ (resp. $\mathcal{R}\mathrm{ind}_{\mathrm{tor}}$). This reduction will be repeated later when proving similar statements.

We first prove the case where u is étale.

1. Let $X_\bullet^+$ be a Čech nerve of f, and $\mathcal{D}_+^{\otimes *\bullet} = {}_{\mathcal{S}\mathrm{ch}^{\mathrm{qc.sep}}}\mathrm{EO}^*_\otimes \circ (X_\bullet^+)^{op}$. By Remark 1.5.5, we only need to check that $\mathcal{D}_+^{*\bullet} = \mathrm{G} \circ \mathcal{D}_+^{\otimes *\bullet}$ is a limit diagram. This is a special case of Lemma 3.1.3 by letting $U_\bullet$ be the sheaf represented by $X_\bullet^+$, and $\mathcal{C}_\bullet$ be the whole category.
2. By [11, 1.3.3], we only need to prove that $\mathcal{D}_+^{!\bullet} = {}_{\mathcal{S}\mathrm{ch}^{\mathrm{qc.sep}}}\mathrm{EO}^! \circ (X_\bullet^+)^{op}$ is a limit diagram. We apply Lemma 3.3.6 below. Assumption (1) follows from the fact that $\mathcal{D}_+^{!-1}$ admits small limits and such limits are preserved by $f^!$. Assumption (2) follows from the Poincaré duality for étale morphisms recalled in Remark 3.3.4. Moreover, $f^!$ is conservative since it is equivalent to f^*.

The general case where u is smooth follows from the above case by Lemma 3.1.2 (3) (and its dual version), and the fact that there exists an étale surjective morphism $g\colon Y \to X$ in $\mathcal{S}\mathrm{ch}^{\mathrm{qc.sep}}$ that factorizes through f [1, 17.16.3 (ii)]. $\square$

Lemma 3.3.6. *Let $\mathcal{C}^\bullet\colon \mathrm{N}(\mathbf{\Delta}_+) \to \mathcal{C}\mathrm{at}_\infty$ be an augmented cosimplicial ∞-category, and set $\mathcal{C} = \mathcal{C}^{-1}$. Let $G\colon \mathcal{C} \to \mathcal{C}^0$ be the evident functor. Assume that:*

1. *The ∞-category $\mathcal{C}^{-1}$ admits limits of G-split cosimplicial objects, and those limits are preserved by G.*
2. *For every morphism $\alpha\colon [m] \to [n]$ in $\mathbf{\Delta}_+$, the diagram*

$$\begin{tikzcd} \mathcal{C}^m \arrow[r, "d^0"] \arrow[d] & \mathcal{C}^{m+1} \arrow[d] \\ \mathcal{C}^n \arrow[r, "d^0"] & \mathcal{C}^{n+1} \end{tikzcd}$$

is right adjointable.

3. *G is conservative.*

Then the canonical map $\theta\colon \mathcal{C} \to \varprojlim_{n\in\mathbf{\Delta}} \mathcal{C}^n$ is an equivalence.

Proof. We only need to apply [30, 6.2.4.3] to the augmented cosimplicial ∞-category $\mathrm{N}(\mathbf{\Delta}_+) \to \mathcal{C}\mathrm{at}_\infty \xrightarrow{R} \mathcal{C}\mathrm{at}_\infty$, where R is the equivalence that associates to every ∞-category its opposite [30, 2.4.2.7]. $\square$

4. DESCENT: A PROGRAM

In this chapter, we develop a program called DESCENT. It is an abstract categorical procedure to extend the maps ${}_{\mathcal{S}\mathrm{ch}^{\mathrm{qc.sep}}}\mathrm{EO}$ (3.5) and ${}_{\mathcal{S}\mathrm{ch}^{\mathrm{qc.sep}}}\mathrm{EO}^*_\otimes$ (3.8) constructed in §3.2 to larger categories. The extended maps satisfy similar properties as the original ones. This program will be run in the next chapter to extend our theory successively to quasi-separated schemes, to algebraic spaces, to Artin stacks, and eventually to higher Deligne–Mumford and higher Artin stacks.

In §4.1, we describe the program by formalizing the data for $\mathcal{S}\mathrm{ch}^{\mathrm{qc.sep}}$. In §4.2, we construct the extension of the maps. In §4.3, we prove the required properties of the extended maps.

4.1. **Description.** In §3.2, we constructed two maps ${}_{\mathcal{S}\mathrm{ch}^{\mathrm{qc.sep}}}\mathrm{EO}$ (3.5) and ${}_{\mathcal{S}\mathrm{ch}^{\mathrm{qc.sep}}}\mathrm{EO}^{*}_{\otimes}$ (3.8). They satisfy certain properties such as descent for smooth morphisms (Proposition 3.3.5). We would like to extend these maps to maps defined on the ∞-category of higher Deligne–Mumford or higher Artin stacks, satisfying similar properties. We will achieve this in many steps, by first extending the maps to quasi-separated schemes, and then to algebraic spaces, and then to Artin stacks, and so on. All the steps are similar to each other. The output of one step provides the input for the next step. We will think of this as recursively running a computer program, which we name DESCENT. In this section, we axiomatize the input and output of this program in an abstract setting.

Let us start with a toy model.

Proposition 4.1.1. *Let $(\tilde{\mathcal{C}}, \tilde{\mathcal{E}})$ be a marked ∞-category such that $\tilde{\mathcal{C}}$ admits pullbacks and $\tilde{\mathcal{E}}$ is stable under composition and pullback. Let $\mathcal{C} \subseteq \tilde{\mathcal{C}}$ be a full subcategory stable under pullback such that for every object X of $\tilde{\mathcal{C}}$, there exists a morphism $Y \to X$ in $\tilde{\mathcal{E}}$ representable in $\mathcal{C}$ with Y in $\mathcal{C}$. Let $\mathcal{D}$ be an ∞-category such that $\mathcal{D}^{op}$ admits geometric realizations. Let $\mathrm{Fun}^{\mathcal{E}}(\mathcal{C}^{op}, \mathcal{D}) \subseteq \mathrm{Fun}(\mathcal{C}^{op}, \mathcal{D})$ (resp. $\mathrm{Fun}^{\tilde{\mathcal{E}}}(\tilde{\mathcal{C}}^{op}, \mathcal{D}) \subseteq \mathrm{Fun}(\tilde{\mathcal{C}}^{op}, \mathcal{D})$) be the full subcategory spanned by functors F such that every edge in $\mathcal{E} = \tilde{\mathcal{E}} \cap \mathcal{C}_1$ (resp. in $\tilde{\mathcal{E}}$) is of F-descent. Then the restriction map*

$$\mathrm{Fun}^{\tilde{\mathcal{E}}}(\tilde{\mathcal{C}}^{op}, \mathcal{D}) \to \mathrm{Fun}^{\mathcal{E}}(\mathcal{C}^{op}, \mathcal{D})$$

is a trivial fibration.

The proof will be given at the end of §4.2.

Example 4.1.2. Let $\mathcal{S}\mathrm{ch}^{\mathrm{qs}} \subseteq \mathcal{S}\mathrm{ch}$ be the full subcategory spanned by quasi-separated schemes. It contains $\mathcal{S}\mathrm{ch}^{\mathrm{qc.sep}}$ as a full subcategory. Applying 4.1.1 to $\tilde{\mathcal{C}} = \mathrm{N}(\mathcal{S}\mathrm{ch}^{\mathrm{qs}})$, $\mathcal{C} = \mathrm{N}(\mathcal{S}\mathrm{ch}^{\mathrm{qc.sep}})$, $\mathcal{D} = \mathcal{P}\mathrm{r}^{\mathrm{L}\otimes}_{\mathrm{st,cl}}$, the set $\tilde{\mathcal{E}}$ of étale surjections and the map ${}_{\mathcal{S}\mathrm{ch}^{\mathrm{qc.sep}}}\mathrm{EO}^{*}_{\otimes}$, we obtain an extension to $\mathrm{N}(\mathcal{S}\mathrm{ch}^{\mathrm{qs}})$.

Now we describe the program in full. We begin by summarizing the categorical properties we need on the geometric side into the following definition.

Definition 4.1.3. An ∞-category $\mathcal{C}$ is *geometric* if it admits small coproducts and pullbacks such that

(1) *Coproducts are disjoint.* Every coCartesian diagram

$$\begin{tikzcd} \emptyset \arrow[r] \arrow[d] & X \arrow[d] \\ Y \arrow[r] & X \coprod Y \end{tikzcd}$$

is also Cartesian, where $\emptyset$ denotes an initial object of $\mathcal{C}$.

(2) *Coproducts are universal.* For a small collection of Cartesian diagrams

$$\begin{tikzcd} Y_i \arrow[r] \arrow[d] & Y \arrow[d] \\ X_i \arrow[r] & X, \end{tikzcd}$$

$i \in I$, the diagram

$$\begin{tikzcd} \coprod_{i\in I} Y_i \arrow[r] \arrow[d] & Y \arrow[d] \\ \coprod_{i\in I} X_i \arrow[r] & X, \end{tikzcd}$$

is also Cartesian.

Remark 4.1.4.

(1) Let $\mathcal{C}$ be geometric. Then a small coproduct of Cartesian diagrams of $\mathcal{C}$ is again Cartesian.
(2) The ∞-categories $\mathrm{N}(\mathcal{S}\mathrm{ch}^{\mathrm{qc.sep}})$, $\mathrm{N}(\mathcal{S}\mathrm{ch}^{\mathrm{qs}})$, $\mathrm{N}(\mathcal{E}\mathrm{sp})$, $\mathrm{N}(\mathcal{C}\mathrm{hp})$, $\mathcal{C}\mathrm{hp}^{k\text{-}\mathrm{Ar}}$ and $\mathcal{C}\mathrm{hp}^{k\text{-}\mathrm{DM}}$ ($k \geq 0$) appearing in this article are all geometric.

We now describe the input and the output of the program. The input has three parts: 0, I, and II. The output has two parts: I and II. We refer the reader to Example 4.1.10 for a typical example.

Input 0. We are given

- A 5-marked ∞-category $(\tilde{\mathcal{C}}, \tilde{\mathcal{E}}_{\mathrm{s}}, \tilde{\mathcal{E}}', \tilde{\mathcal{E}}'', \tilde{\mathcal{E}}_{\mathrm{t}}, \tilde{\mathcal{F}})$, a full subcategory $\mathcal{C} \subseteq \tilde{\mathcal{C}}$, and a morphism $\mathsf{s}'' \to \mathsf{s}'$ of (-1)-truncated objects of $\mathcal{C}$ [29, 5.5.6.1].
- For each $d \in \mathbb{Z} \cup \{-\infty\}$, a subset $\tilde{\mathcal{E}}''_d$ of $\tilde{\mathcal{E}}''$.
- A sequence of inclusions of ∞-categories $\mathcal{L}'' \subseteq \mathcal{L}' \subseteq \mathcal{L}$.
- A function $\dim^+ \colon \tilde{\mathcal{F}} \to \mathbb{Z} \cup \{-\infty, +\infty\}$.

Let $\mathcal{E}_{\mathrm{s}} = \tilde{\mathcal{E}}_{\mathrm{s}} \cap \mathcal{C}_1$, $\mathcal{E}' = \tilde{\mathcal{E}}' \cap \mathcal{C}_1$, $\mathcal{E}'' = \tilde{\mathcal{E}}'' \cap \mathcal{C}_1$, $\mathcal{E}''_d = \tilde{\mathcal{E}}''_d \cap \mathcal{C}_1$ ($d \in \mathbb{Z} \cup \{-\infty\}$), $\mathcal{E}_{\mathrm{t}} = \tilde{\mathcal{E}}_{\mathrm{t}} \cap \mathcal{C}_1$ and $\mathcal{F} = \tilde{\mathcal{F}} \cap \mathcal{C}_1$. Let $\mathcal{C}'$ (resp. $\tilde{\mathcal{C}}'$, $\mathcal{C}''$, and $\tilde{\mathcal{C}}''$) be the full subcategory of $\mathcal{C}$ (resp. $\tilde{\mathcal{C}}$, $\mathcal{C}$, and $\tilde{\mathcal{C}}$) spanned by those objects that admit morphisms to s' (resp. s', s'', and s''). They satisfy

(1) $\tilde{\mathcal{C}}$ is geometric, and the inclusion $\mathcal{C} \subseteq \tilde{\mathcal{C}}$ is stable under finite limits. Moreover, for every small coproduct $X = \coprod_{i \in I} X_i$ in $\tilde{\mathcal{C}}$, X is in $\mathcal{C}$ if and only if X_i is in $\mathcal{C}$ for all $i \in I$.
(2) $\mathcal{L}'' \subseteq \mathcal{L}'$ and $\mathcal{L}' \subseteq \mathcal{L}$ are full subcategories.
(3) $\tilde{\mathcal{E}}_{\mathrm{s}}, \tilde{\mathcal{E}}', \tilde{\mathcal{E}}'', \tilde{\mathcal{E}}_{\mathrm{t}}, \tilde{\mathcal{F}}$ are stable under composition, pullback and small coproducts; and $\tilde{\mathcal{E}}' \subseteq \tilde{\mathcal{E}}'' \subseteq \tilde{\mathcal{E}}_{\mathrm{t}} \subseteq \tilde{\mathcal{F}}$.
(4) For every object X of $\tilde{\mathcal{C}}$, there exists an edge $f \colon Y \to X$ in $\tilde{\mathcal{E}}_{\mathrm{s}} \cap \tilde{\mathcal{E}}'$ with Y in $\mathcal{C}$. Such an f is called an *atlas* for X.
(5) For every object X of $\tilde{\mathcal{C}}$, the diagonal morphism $X \to X \times X$ is representable in $\mathcal{C}$.
(6) For every edge $f \colon Y \to X$ in $\tilde{\mathcal{E}}''$, there exist 2-cells

(4.1)
$$\begin{tikzcd} & Y \arrow[dr, "f"] & \\ Y_d \arrow[ur, "i_d"] \arrow[rr, "f_d"] & & X \end{tikzcd}$$

of $\tilde{\mathcal{C}}$ with f_d in $\tilde{\mathcal{E}}''_d$ such that the edges i_d exhibit Y as the coproduct $\coprod_{d \in \mathbb{Z}} Y_d$.
(7) For every $d \in \mathbb{Z} \cup \{-\infty\}$, $\tilde{\mathcal{E}}''_d \subseteq \tilde{\mathcal{E}}''$ and $\tilde{\mathcal{E}}''_d$ is stable under pullback and small coproducts. $\tilde{\mathcal{E}}''_{-\infty}$ is the set of edges whose source is an initial object. For distinct integers d and e, $\tilde{\mathcal{E}}''_d \cap \tilde{\mathcal{E}}''_e = \tilde{\mathcal{E}}''_{-\infty}$.
(8) For every small set I and every pair of objects X and Y of $\tilde{\mathcal{C}}$, the morphisms $X \to X \coprod Y$ and $\coprod_I X \to X$ are in $\tilde{\mathcal{E}}''_0$. For every 2-cell

(4.2)
$$\begin{tikzcd} & Y \arrow[dr, "f"] & \\ Z \arrow[ur, "g"] \arrow[rr, "h"] & & X \end{tikzcd}$$

of $\tilde{\mathcal{C}}$ with f in $\tilde{\mathcal{E}}''_d$ and g in $\tilde{\mathcal{E}}''_e$, where d and e are integers, h is in $\tilde{\mathcal{E}}''_{d+e}$.
(9) The function $\dim^+$ satisfies the following conditions.
   (a) $\dim^+(f) = -\infty$ if and only if f is in $\tilde{\mathcal{E}}''_{-\infty}$.
   (b) The restriction $\dim^+ |\tilde{\mathcal{E}}''_d - \tilde{\mathcal{E}}''_{-\infty}$ is of constant value d.
   (c) For every 2-cell (4.2) in $\tilde{\mathcal{C}}$ with edges in $\tilde{\mathcal{F}}$, we have $\dim^+(h) \leq \dim^+(f) + \dim^+(g)$, and equality holds when g is in $\tilde{\mathcal{E}}_{\mathrm{s}} \cap \tilde{\mathcal{E}}''$.
   (d) For every Cartesian diagram

$$\begin{tikzcd} W \arrow[r, "g"] \arrow[d, "q"] & Z \arrow[d, "p"] \\ Y \arrow[r, "f"] & X \end{tikzcd}$$

   in $\tilde{\mathcal{C}}$ with f (and hence g) in $\tilde{\mathcal{F}}$, we have $\dim^+(g) \leq \dim^+(f)$, and equality holds when p is in $\tilde{\mathcal{E}}_{\mathrm{s}}$.

(e) For every edge $f\colon Y \to X$ in $\tilde{\mathcal{F}}$ every small collection

$$\begin{tikzcd} & Y \arrow[dr, "f"] & \\ Z_i \arrow[ur, "g_i"] \arrow[rr, "h_i"] & & X \end{tikzcd}$$

of 2-cells with g_i in $\tilde{\mathcal{E}}_{d_i}$ such that the morphism $\coprod_{i\in I} Z_i \to Y$ is in $\tilde{\mathcal{E}}_{\mathrm{s}}$, we have $\dim^+(f) = \sup_{i\in I}\{\dim^+(h_i) - d_i\}$.

(10) $\tilde{\mathcal{E}}' = \tilde{\mathcal{E}}''_0$.

By (6) and (9e), for every small collection $\{Y_i \xrightarrow{f_i} X_i\}_{i\in I}$ of edges in $\tilde{\mathcal{E}}_{\mathrm{t}}$, $\dim^+(\coprod_{i\in I} f_i) = \sup_{i\in I}\{\dim^+(f_i)\}$.

Input I. Input I consists of two maps as follows.

- The *abstract operation map* for $\mathcal{C}'$:

$$_{\mathcal{C}'}\mathrm{EO}\colon \delta^*_{2,\{2\}}\mathrm{Fun}(\Delta^1, \mathcal{C}')^{\mathrm{cart}}_{(\mathcal{F}\cap\mathcal{C}'_1)^0,\mathcal{C}'^{\to}_1} \to \mathrm{Fun}(\mathcal{L}', \mathrm{Mon}^{\mathcal{P}\mathrm{r}^{\mathrm{L}}_{\mathrm{st}}}_{\mathcal{P}\mathrm{f}}(\mathcal{C}\mathrm{at}_\infty)).$$

- $_{\mathcal{C}}\mathrm{EO}^*_\otimes\colon \mathcal{C}^{op} \to \mathrm{Fun}(\mathcal{L}, \mathcal{P}\mathrm{r}^{\mathrm{L}\otimes}_{\mathrm{st,cl}})$.

By definition, s' is a final object of both $\mathcal{C}'$ and $\tilde{\mathcal{C}}'$. We choose a functor $\Sigma\colon \tilde{\mathcal{C}}' \to \mathrm{Fun}(\Delta^1, \tilde{\mathcal{C}}')$ such that $d^1_0 \circ \Sigma = \mathrm{id}_{\tilde{\mathcal{C}}'}$ and $d^1_1 \circ \Sigma$ is the constant functor of value s'. The functor $\Sigma \mid \mathcal{C}'$ induces a map of bisimplicial sets

$$\Sigma_2\colon \mathcal{C}'^{\mathrm{cart}}_{\mathcal{F}\cap\mathcal{C}'_1,\mathcal{C}'_1} \to \mathrm{Fun}(\Delta^1, \mathcal{C}')^{\mathrm{cart}}_{(\mathcal{F}\cap\mathcal{C}'_1)^0,\mathcal{C}'^{\to}_1}.$$

We call the following composite map

$$(4.3)\quad {}_{\mathcal{C}'}\mathrm{EO}^*_!\colon \delta^*_{2,\{2\}}\mathcal{C}'^{\mathrm{cart}}_{\mathcal{F}\cap\mathcal{C}'_1,\mathcal{C}'_1} \xrightarrow{\delta^*_{2,\{2\}}\Sigma_2} \delta^*_{2,\{2\}}\mathrm{Fun}(\Delta^1, \mathcal{C}')^{\mathrm{cart}}_{(\mathcal{F}\cap\mathcal{C}'_1)^0,\mathcal{C}'^{\to}_1} \xrightarrow{_{\mathcal{C}'}\mathrm{EO}} \mathrm{Fun}(\mathcal{L}', \mathrm{Mon}^{\mathcal{P}\mathrm{r}^{\mathrm{L}}_{\mathrm{st}}}_{\mathcal{P}\mathrm{f}}(\mathcal{C}\mathrm{at}_\infty)) \xrightarrow{\mathrm{Fun}(\mathcal{L}', \mathrm{G}_{(\langle 1\rangle,\{1\})})} \mathcal{P}\mathrm{r}^{\mathrm{L}}_{\mathrm{st}}$$

the *abstract Base Change map* for $\mathcal{C}'$. Restricting (4.3) to the first direction, we get

$$_{\mathcal{C}'}\mathrm{EO}_!\colon \mathcal{C}' \to \mathrm{Fun}(\mathcal{L}', \mathcal{P}\mathrm{r}^{\mathrm{L}}_{\mathrm{st}}).$$

Input I is subject to the following properties:

P1: *Disjointness.* The map $_{\mathcal{C}}\mathrm{EO}^*_\otimes$ sends small coproducts to products.

P2: *Compatibility.* The map $\mathrm{Fun}(\mathcal{L}', \mathrm{pf}) \circ \mathrm{Fun}((\Delta^1)^{op}, ({}_{\mathcal{C}}\mathrm{EO}^*_\otimes \mid \mathcal{C}'^{op}))$ is equivalent to

$$(4.4)\quad {}_{\mathcal{C}'}\mathrm{EO}^*_{\mathrm{pf}}\colon \mathrm{Fun}(\Delta^1, \mathcal{C}')^{op} \to \delta^*_{2,\{2\}}\mathrm{Fun}(\Delta^1, \mathcal{C}')^{\mathrm{cart}}_{(\mathcal{F}\cap\mathcal{C}'_1)^0,\mathcal{C}'^{\to}_1} \xrightarrow{_{\mathcal{C}'}\mathrm{EO}} \mathrm{Fun}(\mathcal{L}', \mathrm{Mon}^{\mathcal{P}\mathrm{r}^{\mathrm{L}}_{\mathrm{st}}}_{\mathcal{P}\mathrm{f}}(\mathcal{C}\mathrm{at}_\infty)),$$

where the first map is the restriction to direction 2.

At this point we fix some notations. For an object X of $\mathcal{C}$ and λ of $\mathcal{L}$, we denote by $\mathcal{D}(X,\lambda)^\otimes$ the symmetric monoidal ∞-category $_{\mathcal{C}}\mathrm{EO}^*_\otimes(X)(\lambda)$. For a morphism $f\colon Y \to X$ of $\mathcal{C}$ (resp. $\mathcal{C}'$) and an object λ of $\mathcal{L}$ (resp. $\mathcal{L}'$), we denote by $f^*\colon \mathcal{D}(X,\lambda)^\otimes \to \mathcal{D}(Y,\lambda)^\otimes$ (resp. $f_!\colon \mathcal{D}(Y,\lambda) \to \mathcal{D}(X,\lambda)$) the functor $_{\mathcal{C}}\mathrm{EO}^*_\otimes(f)(\lambda)$ (resp. $_{\mathcal{C}'}\mathrm{EO}_!(f)(\lambda)$). By (P2), $\mathcal{D}(X,\lambda)$ is equivalent to the underlying ∞-category of $\mathcal{D}(X,\lambda)^\otimes$, which justifies the notation.

P3: *Conservativeness.* If f is in $\mathcal{E}_{\mathrm{s}}$, then f^* is conservative.

P4: *Descent.* Let f be a morphism of $\mathcal{C}$. Then f is of universal $_{\mathcal{C}}\mathrm{EO}^*_\otimes$-descent (resp. $_{\mathcal{C}'}\mathrm{EO}_!$-codescent) if f is in $\mathcal{E}_{\mathrm{s}} \cap \mathcal{E}''$ (resp. $\mathcal{E}_{\mathrm{s}} \cap \mathcal{E}'' \cap \mathcal{C}'_1$).

P5: *Adjointability for $\mathcal{E}'$.* Let

$$\begin{tikzcd} W \arrow[r, "g"] \arrow[d, "q"'] & Z \arrow[d, "p"] \\ Y \arrow[r, "f"] & X \end{tikzcd}$$

be a Cartesian diagram of $\mathcal{C}'$ with f in $\mathcal{E}'$, and λ be an object of $\mathcal{L}'$. Then

(1) The square

$$\begin{tikzcd} \mathcal{D}(Z,\lambda) \arrow[d, "g^*"'] & \mathcal{D}(X,\lambda) \arrow[l, "p^*"'] \arrow[d, "f^*"] \\ \mathcal{D}(W,\lambda) & \mathcal{D}(Y,\lambda) \arrow[l, "q^*"'] \end{tikzcd}$$

has a right adjoint that is a square of $\mathcal{P}\mathrm{r}^{\mathrm{R}}_{\mathrm{st}}$.

(2) If p is also in $\mathcal{E}'$, the square

$$\begin{tikzcd} \mathcal{D}(X,\lambda) \arrow[d, "p^*"'] & \mathcal{D}(Y,\lambda) \arrow[l, "f_!"'] \arrow[d, "q^*"] \\ \mathcal{D}(Z,\lambda) & \mathcal{D}(W,\lambda) \arrow[l, "g_!"'] \end{tikzcd}$$

is right adjointable.

P5$^{\mathbf{bis}}$: *Adjointability for $\mathcal{E}''$.* We have the same statement as in (P4) after replacing $\mathcal{C}'$ by $\mathcal{C}''$, $\mathcal{E}'$ by $\mathcal{E}''$, and $\mathcal{L}'$ by $\mathcal{L}''$.

The validity of the axioms is independent of the choice of Σ.

Input II. Input II consists of the following data.

- A section $\mathrm{ST} = (\lambda_X\langle d\rangle)_{d\in\mathbb{Z},(X,\lambda)\in\mathcal{C}''^{op}\times\mathcal{L}''}$ of $\mathrm{Un}_{\mathcal{C}''\times\mathcal{L}''^{op}}({}_{\mathcal{C}''}\mathrm{EO}^*_{\mathrm{ST}})$, where the notations are similarly defined as in Definition 3.3.1.
- A t-structure on $\mathcal{D}(X,\lambda)$ for every object X of $\mathcal{C}$ and every object λ of $\mathcal{L}$.
- A morphism (the *trace map* for $\mathcal{E}_{\mathrm{t}}$) $\mathrm{Tr}_f\colon \tau^{\geq 0} f_!\lambda_Y\langle d\rangle \to \lambda_X$ for every edge $f\colon Y\to X$ in $\mathcal{E}_{\mathrm{t}}\cap\mathcal{C}''_1$, every integer $d\geq \dim^+(f)$, and every object λ of $\mathcal{L}''$.
- A morphism (the *trace map* for $\mathcal{E}'$) $\mathrm{Tr}_f\colon \tau^{\geq 0} f_!\lambda_Y \to \lambda_X$ for every edge $f\colon Y\to X$ in $\mathcal{E}'\cap\mathcal{C}'_1$ and every object λ of $\mathcal{L}'$. These trace maps coincide when both are defined.

Input II is subject to the following properties.

P6: *t-structure.* For every object λ of $\mathcal{L}$, we have the following.

(1) For every object X of $\mathcal{C}$, λ_X is in the heart $\mathcal{D}^\heartsuit(X,\lambda)$ of $\mathcal{D}(X,\lambda)$, and $-\otimes\lambda_X\langle 1\rangle$ is t-exact if it is defined.

(2) For every object X of $\mathcal{C}$, the t-structure on $\mathcal{D}(X,\lambda)$ is accessible, right complete, and $\mathcal{D}^{\leq-\infty}(X,\lambda) := \bigcap_n \mathcal{D}^{\leq -n}(X,\lambda)$ consists of zero objects.

(3) For every morphism f of $\mathcal{C}$, f^* is t-exact.

P7: *Poincaré duality for $\mathcal{E}''$.* We have

(1) For every f in $\mathcal{E}_{\mathrm{t}}\cap\mathcal{C}''_1$, every integer $d\geq\dim^+(f)$, and every object λ of $\mathcal{L}''$, the source of the trace map Tr_f belongs to the heart $\mathcal{D}^\heartsuit(X,\lambda)$. Moreover, Tr_f is functorial in the sense of Remark 3.3.2 with $\mathrm{N}(\mathcal{S}\mathrm{ch}^{\mathrm{qc.sep}})$ (resp. $\mathrm{N}(\mathcal{R}\mathrm{ind}^{op}_{\text{L-tor}})$) replaced by $\mathcal{C}''$ (resp. $\mathcal{L}''$).

(2) For every f in $\mathcal{E}''_d\cap\mathcal{C}''_1$, and every object λ of $\mathcal{L}''$, the map $u_f\colon f_!\circ f^*\langle d\rangle\to\mathrm{id}_X$, induced by the trace map $\mathrm{Tr}_f\colon\tau^{\geq 0}f_!\lambda_Y\langle d\rangle\to\lambda_X$ following the procedure in Remark 3.3.3, is a counit transformation. Here id_X is the identity functor of $\mathcal{D}(X,\lambda)$.

P7$^{\mathbf{bis}}$: *Poincaré duality for $\mathcal{E}'$.* We have the same statement as in (P7) after letting $d=0$, and replacing $\mathcal{C}''$ by $\mathcal{C}'$, $\mathcal{E}_{\mathrm{t}}$ by $\mathcal{E}'$, and $\mathcal{L}''$ by $\mathcal{L}'$.

Remark 4.1.5.

(1) (P4) implies that (P3) holds for $f\in\mathcal{E}_{\mathrm{s}}\cap\mathcal{E}''$.

(2) If $d>\dim^+(f)$, the trace map Tr_f is not interesting because its source $\tau^{\geq 0}f_!\lambda_Y\langle d\rangle$ is a zero object. We included such maps in the data in order to state the functoriality (in the sense of Remark 3.3.2) more conveniently.

(3) We extend the trace map to morphisms $f\colon Y\to X$ in $\mathcal{E}_{\mathrm{t}}\cap\mathcal{C}''_1$ endowed with 2-cells (4.1) satisfying $\dim^+(f_d)\leq d$ and such that the morphisms i_d exhibit Y as $\coprod_{d\in\mathbb{Z}} Y_d$. For every object λ of $\mathcal{L}''$,

the map
$$\mathcal{D}(Y,\lambda) \to \prod_{d\in\mathbb{Z}} \mathcal{D}(Y_d,\lambda),$$
induced by i_d is an equivalence by (P1). We write $-\langle \dim^+\rangle\colon \mathcal{D}(Y,\lambda) \to \mathcal{D}(Y,\lambda)$ for the product of $(-\langle d\rangle\colon \mathcal{D}(Y_d,\lambda) \to \mathcal{D}(Y_d,\lambda))_{d\in\mathbb{Z}}$. Since $\lambda_Y \simeq \bigoplus_{d\in\mathbb{Z}} i_{d!}\lambda_{Y_d}$, the maps Tr_{f_d} induce a map $\mathrm{Tr}_f\colon \tau^{\geq 0} f_!\lambda_Y\langle \dim^+\rangle \to \lambda_X$. Moreover, the trace map is functorial in the sense that an analogue of Remark 3.3.2 holds.

(4) (P7) (2) still holds for morphisms $f\colon Y \to X$ in $\mathcal{E}''\cap\mathcal{C}''_1$. For such morphisms, the 2-cells in Input 0 (6) are unique up to equivalence by Input 0 (7). We write $-\langle \dim f\rangle\colon \mathcal{D}(Y,\lambda) \to \mathcal{D}(Y,\lambda)$ for the product of $(-\langle d\rangle\colon \mathcal{D}(Y_d,\lambda) \to \mathcal{D}(Y_d,\lambda))_{d\in\mathbb{Z}}$. Then, (P7) (2) for the morphisms f_d implies that the map $u_f\colon f_! \circ f^*\langle \dim f\rangle \to \mathrm{id}_X$ induced by the trace map $\mathrm{Tr}_f\colon \tau^{\geq 0} f_!\lambda_Y\langle d\rangle \to \lambda_X$ following the procedure in Remark 3.3.3 is a counit transformation.

Output I. Output I consists of the following two maps.

- The *abstract operation map* for $\tilde{\mathcal{C}}'$:
$$_{\tilde{\mathcal{C}}'}\mathrm{EO}\colon \delta^*_{2,\{2\}}\mathrm{Fun}(\Delta^1,\tilde{\mathcal{C}}')^{\mathrm{cart}}_{(\tilde{\mathcal{F}}\cap\tilde{\mathcal{C}}'_1)^0,\tilde{\mathcal{C}}'^{\to}_1} \to \mathrm{Fun}(\mathcal{L}',\mathrm{Mon}^{\mathcal{P}\mathrm{r}^{\mathrm{L}}_{\mathrm{st}}}_{\mathcal{P}\mathrm{f}}(\mathcal{C}\mathrm{at}_\infty))$$
extending $_{\mathcal{C}'}\mathrm{EO}$.
- $_{\tilde{\mathcal{C}}}\mathrm{EO}^*_\otimes\colon \tilde{\mathcal{C}}^{op} \to \mathrm{Fun}(\mathcal{L},\mathcal{P}\mathrm{r}^{\mathrm{L}\otimes}_{\mathrm{st,cl}})$ extending $_{\mathcal{C}}\mathrm{EO}^*_\otimes$.

Output II. Output II consists of the following data extending those of Input II.

- A section $\mathrm{ST} = (\lambda_X\langle d\rangle)_{d\in\mathbb{Z},(X,\lambda)\in\tilde{\mathcal{C}}''^{op}\times\mathcal{L}''}$ of $\mathrm{Un}_{\tilde{\mathcal{C}}''\times\mathcal{L}''^{op}}(_{\tilde{\mathcal{C}}''}\mathrm{EO}^*_{\mathrm{ST}})$.
- A t-structure on $\mathcal{D}(X,\lambda)$ for every object X of $\tilde{\mathcal{C}}$ and every object λ of $\mathcal{L}$.
- A morphism (the *trace map* for $\tilde{\mathcal{E}}_{\mathrm{t}}$) $\mathrm{Tr}_f\colon \tau^{\geq 0} f_!\lambda_Y\langle d\rangle \to \lambda_X$ for every edge $f\colon Y \to X$ in $\tilde{\mathcal{E}}_{\mathrm{t}}\cap\tilde{\mathcal{C}}''_1$, every integer $d \geq \dim^+(f)$, and every object λ of $\mathcal{L}''$.
- A morphism (the *trace map* for $\tilde{\mathcal{E}}'$) $\mathrm{Tr}_f\colon \tau^{\geq 0} f_!\lambda_Y \to \lambda_X$ for every edge $f\colon Y \to X$ in $\tilde{\mathcal{E}}'\cap\tilde{\mathcal{C}}'_1$ and every object λ of $\mathcal{L}'$. These trace maps coincide when both are defined.

We define properties (P1) through (P7$^{\mathrm{bis}}$) for Output I and II by replacing $\mathcal{C}'$, $\mathcal{C}''$ and $(\mathcal{C},\mathcal{E}_{\mathrm{s}},\mathcal{E}',\mathcal{E}'',\mathcal{E}_{\mathrm{t}},\mathcal{F})$ by $\tilde{\mathcal{C}}'$, $\tilde{\mathcal{C}}''$ and $(\tilde{\mathcal{C}},\tilde{\mathcal{E}}_{\mathrm{s}},\tilde{\mathcal{E}}',\tilde{\mathcal{E}}'',\tilde{\mathcal{E}}_{\mathrm{t}},\tilde{\mathcal{F}})$, respectively.

Theorem 4.1.6. *Fix an Input 0. Then*

(1) *Every Input I satisfying (P1) through (P5^{bis}) can be extended to an Output I satisfying (P1) through (P5^{bis}).*

(2) *For given Input I, II satisfying (P1) through (P7^{bis}) and given Output I extending Input I and satisfying (P1) through (P5^{bis}), there exists an Output II extending Input II and satisfying (P6), (P7), (P7^{bis}).*

Output I will be accomplished in §4.2. Output II and the proof of properties (P1) through (P7$^{\mathrm{bis}}$) will be accomplished in §4.3.

Variant 4.1.7. Let us introduce a variant of DESCENT. In Input 0, we let $\tilde{\mathcal{E}}' = \tilde{\mathcal{E}}''$, $\mathsf{s}' \to \mathsf{s}''$ be a degenerate edge, $\mathcal{L}' = \mathcal{L}''$, and *ignore* (10). In Input II, we also *ignore* the trace map for $\mathcal{E}'$ and property (P7$^{\mathrm{bis}}$). In particular, (P5) and (P5$^{\mathrm{bis}}$) coincide. Theorem 4.1.6 for this variant still holds and will be applied to (higher) Artin stacks.

Remark 4.1.8.

(1) If the only goal is to extend $_{\mathcal{C}}\mathrm{EO}$ and $_{\mathcal{C}}\mathrm{EO}^*_\otimes$, the statement of Theorem 4.1.6 (1) can be made more compact: every Input I satisfying properties (P2), (P4), and (P5) can be extended to an Output I satisfying (P2), (P4), and (P5). This will follow from our proof of Theorem 4.1.6 in this chapter.

(2) The Output I in Theorem 4.1.6 (1) is unique up to equivalence. More precisely, we can define a simplicial set K classifying those Input I that satisfy (P2) and (P4). The 0-cells of K are triples $(_{\mathcal{C}}\mathrm{EO}, {}_{\mathcal{C}}\mathrm{EO}^*_\otimes, h)$, where h is the equivalence in (P2). Similarly, let K' be the simplicial set classifying those Output II that satisfy (P2) and (P4). Then the restriction map $K' \to K$

satisfies the right lifting property with respect to $\partial\Delta^n \subseteq \Delta^n$ for all $n \geq 1$. One can show this by adapting our proof of Theorem 4.1.6. Moreover, in all the above, h can be taken to be the identity without loss of generality.

(3) The Output II in Theorem 4.1.6 (2) is also unique up to equivalence. More precisely, let us fix an Output I extending Input I and satisfying (P2) and (P4). Since $\mathcal{C}'' \subseteq \tilde{\mathcal{C}}''$ is right anodyne, the restriction map

$$\mathrm{Map}_{\tilde{\mathcal{C}}''\times\mathcal{L}''^{op}}(\tilde{\mathcal{C}}''\times\mathcal{L}''^{op}, \mathrm{Un}_{\tilde{\mathcal{C}}''\times\mathcal{L}''^{op}}({}_{\tilde{\mathcal{C}}''}\mathrm{EO}^*_{\mathrm{ST}})) \to \mathrm{Map}_{\mathcal{C}''\times\mathcal{L}''^{op}}(\mathcal{C}''\times\mathcal{L}''^{op}, \mathrm{Un}_{\mathcal{C}''\times\mathcal{L}''^{op}}({}_{\mathcal{C}''}\mathrm{EO}^*_{\mathrm{ST}}))$$

is a trivial fibration. Fix a section of $\mathrm{Un}_{\tilde{\mathcal{C}}''\times\mathcal{L}''^{op}}({}_{\tilde{\mathcal{C}}''}\mathrm{EO}^*_{\mathrm{ST}})$ extending the original one, and an assignment of t-structures for the Input satisfying (P6). Then there exists a unique extension to the Output satisfying (P6). Moreover, for every assignment of traces for the Input satisfying (P7) (resp. (P7$^{\mathrm{bis}}$)), there exists a unique extension to the Output satisfying (P7) (resp. (P7$^{\mathrm{bis}}$)). Note that the trace map is defined in the heart, so that no homotopy issue arises.

Definition 4.1.9. For a morphism $f\colon Y \to X$ locally of finite type between algebraic spaces, we define the *upper relative dimension* of f to be $\sup\{\dim(Y \times_X \operatorname{Spec}\Omega)\} \in \mathbb{Z}\cup\{-\infty,+\infty\}$ [4, 04N6], where the supremum is taken over all geometric points $\operatorname{Spec}\Omega \to X$. We adopt the convention that the empty scheme has dimension $-\infty$.

Example 4.1.10. The initial input for DESCENT is the following:

- $\tilde{\mathcal{C}} = \mathrm{N}(\mathcal{S}\mathrm{ch}^{\mathrm{qs}})$. It is geometric and admits $\operatorname{Spec}\mathbb{Z}$ as a final object.
- $\mathcal{C} = \mathrm{N}(\mathcal{S}\mathrm{ch}^{\mathrm{qc.sep}})$, and $\mathsf{s}' \to \mathsf{s}''$ is the unique morphism $\operatorname{Spec}\mathbb{Z}[\mathsf{L}^{-1}] \to \operatorname{Spec}\mathbb{Z}$. In particular, $\mathcal{C}' = \mathcal{C}$ and $\tilde{\mathcal{C}}' = \tilde{\mathcal{C}}$.
- $\tilde{\mathcal{E}}_{\mathrm{s}}$ is the set of *surjective* morphisms.
- $\tilde{\mathcal{E}}'$ is the set of *étale* morphisms.
- $\tilde{\mathcal{E}}''$ is the set of *smooth* morphisms.
- $\tilde{\mathcal{E}}''_d$ is the set of *smooth* morphisms of pure relative dimension d.
- $\tilde{\mathcal{E}}_{\mathrm{t}}$ is the set of morphisms that are *flat and locally of finite presentation*.
- $\tilde{\mathcal{F}}$ is the set of morphisms *locally of finite type*.
- $\mathcal{L} = \mathrm{N}(\mathcal{R}\mathrm{ind}^{op})$, $\mathcal{L}' = \mathrm{N}(\mathcal{R}\mathrm{ind}^{op}_{\mathrm{tor}})$, and $\mathcal{L}'' = \mathrm{N}(\mathcal{R}\mathrm{ind}^{op}_{\mathsf{L}\text{-}\mathrm{tor}})$.
- $\dim^+$ is the (function of) upper relative dimension (Definition 4.1.9).
- ${}_{\mathcal{C}'}\mathrm{EO}$ is (3.5) (in its equivalent form), and ${}_{\mathcal{C}}\mathrm{EO}^*_{\otimes}$ is (3.8).
- $\mathrm{ST} = \mathrm{S}_2\mathrm{T}_1$ is defined in Definition 3.3.1.
- $\mathcal{D}(X,\lambda)$ is endowed with its usual t-structure recalled in Definition 3.3.1.
- The trace maps are the classical ones as recalled in Remarks 3.3.2 and 3.3.4.

The properties (P1) through (P7$^{\mathrm{bis}}$) are satisfied:

(P1) This is Lemma 2.2.5.
(P2) This follows from our construction. In fact, the two maps are equal in this case.
(P3) This is obvious.
(P4) This is Proposition 3.3.5.
(P5) This follows from Lemma 4.1.11 below. Part (1) of (P5), namely étale base change, is trivial.
(P5$^{\mathrm{bis}}$) This follows from Lemma 4.1.11. Part (1) of (P5$^{\mathrm{bis}}$) is smooth base change.
(P6) (1) follows from [30, 1.3.4.21]. (2) and (3) are obvious.
(P7) This has been recalled in Remarks 3.3.2 and 3.3.3.
(P7$^{\mathrm{bis}}$) This has been recalled in Remark 3.3.4.

Lemma 4.1.11. *Assume (P7). Then (P5) holds. Moreover, part (2) of (P5) holds without the assumption that p is also in $\mathcal{E}'$.*

Similarly, (P7$^{\mathrm{bis}}$) implies that (P5$^{\mathrm{bis}}$) holds without the assumption that p is also in $\mathcal{E}''$.

Proof. We denote by p_* (resp. q_*) a right adjoint of p^* (resp. q^*) and by $f^!$ (resp. $g^!$) a right adjoint of $f_!$ (resp. $g_!$).

(1) By (P7), f^* and g^* have left adjoints. Moreover, the diagram

$$
\begin{tikzcd}
f^*p_*\langle \dim f\rangle \arrow[r] \arrow[d] \arrow[dd, bend right=60, "\simeq"'] & q_*g^*\langle \dim f\rangle \arrow[rr, equal] \arrow[d] & & q_*g^*\langle \dim f\rangle \arrow[d] \arrow[dd, bend left=60, "\simeq"] \\
f^!f_!f^*p_*\langle \dim f\rangle \arrow[r] \arrow[d, "\mathrm{Tr}_f"] & f^!f_!q_*g^*\langle \dim f\rangle \arrow[r] & f^!p_*g_!g^*\langle \dim f\rangle \arrow[r, "\sim"] \arrow[d, "\mathrm{Tr}_g"'] & q_*g^!g_!g^*\langle \dim f\rangle \arrow[d, "\mathrm{Tr}_g"'] \\
f^!p_* \arrow[rr, equal] & & f^!p_* \arrow[r, "\sim"] & q_*g^!
\end{tikzcd} \tag{4.5}
$$

is commutative up to homotopy. It follows that the top horizontal arrow is an equivalence.

(2) Since the diagram

$$
\begin{tikzcd}
q^*f^*\langle \dim f\rangle \arrow[rr, equal] \arrow[d] \arrow[dd, bend right=60, "\simeq"'] & & q^*f^*\langle \dim f\rangle \arrow[r, "\sim"] & g^*p^*\langle \dim f\rangle \arrow[d] \arrow[dd, bend left=60, "\simeq"] \\
q^*f^!f_!f^*\langle \dim f\rangle \arrow[r] \arrow[d, "\mathrm{Tr}_f"'] & g^!p^*f_!f^*\langle \dim f\rangle \arrow[r, "\sim"] \arrow[d, "\mathrm{Tr}_f"] & g^!g_!q^*f^*\langle \dim f\rangle \arrow[r, "\sim"] & g^!g_!g^*p^*\langle \dim f\rangle \arrow[d, "\mathrm{Tr}_g"'] \\
q^*f^! \arrow[r] & g^!p^* \arrow[rr, equal] & & g^!p^*
\end{tikzcd}
$$

is commutative up to homotopy, the bottom horizontal arrow is an equivalence.

$\square$

4.2. **Construction.** The goal of this subsection is to construct the maps ${}_{\tilde{\mathcal{C}}'}\mathrm{EO}$ and ${}_{\tilde{\mathcal{C}}}\mathrm{EO}^*_{\otimes}$ of Output I in §4.1. We will construct Output II and check the nine properties (P1) – (P7^{bis}) in the next section.

Let us start from the construction of ${}_{\tilde{\mathcal{C}}'}\mathrm{EO}$. Let $\mathcal{R} \subseteq \tilde{\mathcal{F}} \cap \tilde{\mathcal{C}}'$ be the set of morphisms that are representable in $\mathcal{C}'$. We have successive inclusions

$$\mathrm{Fun}(\Delta^1, \mathcal{C}')^{\mathrm{cart}}_{(\mathcal{F}\cap\mathcal{C}'_1)^0, \mathcal{C}'^{\to}_1} \subseteq \mathrm{Fun}(\Delta^1, \tilde{\mathcal{C}}')^{\mathrm{cart}}_{\mathcal{R}^0, \tilde{\mathcal{C}}'^{\to}_1} \subseteq \mathrm{Fun}(\Delta^1, \tilde{\mathcal{C}}')^{\mathrm{cart}}_{(\tilde{\mathcal{F}}\cap\tilde{\mathcal{C}}'_1)^0, \tilde{\mathcal{C}}'^{\to}_1}. \tag{4.6}$$

We proceed in two steps. The first step extends ${}_{\mathcal{C}'}\mathrm{EO}$ to the map ${}^{\mathcal{R}}_{\tilde{\mathcal{C}}'}\mathrm{EO}$ with the source $\delta^*_{2,\{2\}}\mathrm{Fun}(\Delta^1, \tilde{\mathcal{C}}')^{\mathrm{cart}}_{\mathcal{R}^0, \tilde{\mathcal{C}}'^{\to}_1}$.

Step 1. An n-cell σ_n of $\delta^*_{2,\{2\}}\mathrm{Fun}(\Delta^1, \tilde{\mathcal{C}}')^{\mathrm{cart}}_{\mathcal{R}^0, \tilde{\mathcal{C}}'^{\to}_1}$ is given by a functor $\sigma\colon \Delta^n \times (\Delta^n)^{op} \to \mathrm{Fun}(\Delta^1, \tilde{\mathcal{C}}')$. We define $\mathrm{Cov}(\sigma)$ to be the full subcategory of

$$\mathrm{Fun}(\Delta^n \times (\Delta^n)^{op} \times \mathrm{N}(\mathbf{\Delta}^{op}_+), \mathrm{Fun}(\Delta^1, \tilde{\mathcal{C}}')) \times_{\mathrm{Fun}(\Delta^n \times (\Delta^n)^{op} \times \{\emptyset\}, \mathrm{Fun}(\Delta^1, \tilde{\mathcal{C}}'))} \{\sigma\}$$

spanned by functors $\sigma^0\colon \Delta^n \times (\Delta^n)^{op} \times \mathrm{N}(\mathbf{\Delta}^{op}_+) \to \mathrm{Fun}(\Delta^1, \tilde{\mathcal{C}}')$ such that

- for every objects (i,j) of $\Delta^n \times (\Delta^m)^{op}$, the restriction $\sigma^0 \mid \Delta^{\{(i,j)\}} \times \mathrm{N}((\mathbf{\Delta}^{\le 0}_+)^{op})$ is given by a square

$$
\begin{tikzcd}
Y^{i,j}_0 \arrow[r] \arrow[d, "g^{i,j}"'] & X^{i,j}_0 \arrow[d, "f^{i,j}"] \\
Y^{i,j}_{-1} \arrow[r] & X^{i,j}_{-1}
\end{tikzcd} \tag{4.7}
$$

  where $f^{i,j}$ and $g^{i,j}$ are atlases;
- σ is a right Kan extension of $\sigma^0 \mid \Delta^{\{n\}} \times (\Delta^n)^{op} \times (\mathbf{\Delta}^{\le 0}_+)^{op} \cup \Delta^n \times (\Delta^n)^{op} \times \{\emptyset\}$.

In particular, objects σ^0 of $\mathrm{Cov}(\sigma)$ satisfy

- for every object (i,j) of $\Delta^n \times (\Delta^n)^{op}$, $\sigma^0 \mid \Delta^{\{(i,j)\}} \times \mathrm{N}(\mathbf{\Delta}^{op}_+)$ is a Čech nerve of (4.7).

The ∞-category $\mathrm{Cov}(\sigma)$ is nonempty by Input 0 (4) and (5), and admits product of two objects. Indeed, for every pair of objects σ_1^0 and σ_2^0 of $\mathrm{Cov}(\sigma)$,

$$(\sigma_1^0 \times \sigma_2^0)(i,j,[k]) \simeq \sigma_1^0(i,j,[k]) \times_{\sigma(i,j)} \sigma_2^0(i,j,[k]).$$

Therefore, by Lemma 1.1.1, $\mathrm{Cov}(\sigma)$ is a weakly contractible Kan complex.

The restriction functor

$$\mathrm{Cov}(\sigma_n) \to \mathrm{Fun}(\mathrm{N}(\mathbf{\Delta}^{op}) \times \Delta^n \times (\Delta^n)^{op}, \mathrm{Fun}(\Delta^1, \mathcal{C}')).$$

induces a map

$$\mathrm{Cov}(\sigma_n)^{op} \to \mathrm{Fun}(\mathrm{N}(\mathbf{\Delta}), \mathrm{Fun}(\Delta^n, \delta^*_{2,\{2\}}\mathrm{Fun}(\Delta^1, \mathcal{C}')^{\mathrm{cart}}_{(\mathcal{F}'\cap\mathcal{C}'_1)^0, \mathcal{C}'^{\to}_1})).$$

Composing with the map ${}_{\mathcal{C}'}\mathrm{EO}$, we obtain a map

$$\phi(\sigma_n)\colon \mathrm{Cov}(\sigma_n)^{op} \to \mathrm{Fun}(\mathrm{N}(\mathbf{\Delta}), \mathrm{Fun}(\Delta^n, \mathrm{Fun}(\mathcal{L}', \mathrm{Mon}_{\mathcal{P}\mathrm{f}}^{\mathcal{P}\mathrm{r}^{\mathrm{L}}_{\mathrm{st}}}(\mathcal{C}\mathrm{at}_\infty)))).$$

Let $\mathcal{K} \subseteq \mathrm{Fun}(\mathrm{N}(\mathbf{\Delta}_+), \mathrm{Fun}(\Delta^n, \mathrm{Fun}(\mathcal{L}', \mathrm{Mon}_{\mathcal{P}\mathrm{f}}^{\mathcal{P}\mathrm{r}^{\mathrm{L}}_{\mathrm{st}}}(\mathcal{C}\mathrm{at}_\infty))))$ be the full subcategory spanned by those functors $F\colon \mathrm{N}(\mathbf{\Delta}_+) \to \mathrm{Fun}(\Delta^n, \mathrm{Fun}(\mathcal{L}', \mathrm{Mon}_{\mathcal{P}\mathrm{f}}^{\mathcal{P}\mathrm{r}^{\mathrm{L}}_{\mathrm{st}}}(\mathcal{C}\mathrm{at}_\infty)))$ which are right Kan extensions of $F \mid \mathrm{N}(\mathbf{\Delta})$. Consider the following diagram

$$\begin{tikzcd}
\mathcal{N}(\sigma_n) \arrow[r] \arrow[d, "\mathrm{res}_1^*\phi(\sigma_n)"'] & \mathrm{Cov}(\sigma_n)^{op} \arrow[d, "\phi(\sigma_n)"] \\
\mathcal{K} \arrow[r, "\mathrm{res}_1"] \arrow[d, "\mathrm{res}_2"'] & \mathrm{Fun}(\mathrm{N}(\mathbf{\Delta}), \mathrm{Fun}(\Delta^n, \mathrm{Fun}(\mathcal{L}', \mathrm{Mon}_{\mathcal{P}\mathrm{f}}^{\mathcal{P}\mathrm{r}^{\mathrm{L}}_{\mathrm{st}}}(\mathcal{C}\mathrm{at}_\infty)))) \\
\mathrm{Fun}(\Delta^n, \mathrm{Fun}(\mathcal{L}', \mathrm{Mon}_{\mathcal{P}\mathrm{f}}^{\mathcal{P}\mathrm{r}^{\mathrm{L}}_{\mathrm{st}}}(\mathcal{C}\mathrm{at}_\infty))), &
\end{tikzcd}$$

where the upper square is Cartesian, and res_2 is the restriction to $\{\emptyset\}$. Let $\Phi(\sigma_n) = \mathrm{res}_2 \circ \mathrm{res}_1^*\phi(\sigma_n)$. It is easy to see that the above process is functorial so that the collection of $\Phi(\sigma_n)$ defines a morphism Φ in the category $(\mathcal{S}\mathrm{et}_\Delta)^{(\mathbf{\Delta}_{\delta^*_{2,\{2\}}\mathrm{Fun}(\Delta^1, \tilde{\mathcal{C}}')^{\mathrm{cart}}_{\mathcal{R}^0, \tilde{\mathcal{C}}'^{\to}_1}})^{op}}$.

Lemma 4.2.1. *The map $\Phi(\sigma_n)$ takes values in* $\mathrm{Map}^\sharp((\Delta^n)^\flat, \mathrm{Fun}(\mathcal{L}', \mathrm{Mon}_{\mathcal{P}\mathrm{f}}^{\mathcal{P}\mathrm{r}^{\mathrm{L}}_{\mathrm{st}}}(\mathcal{C}\mathrm{at}_\infty))^\natural)$.

Let X_{-1} be an object of $\mathcal{C}'$ and let $\mathrm{Cov}(X_{-1})$ be the full subcategory of

$$\mathrm{Fun}(\mathrm{N}((\mathbf{\Delta}_+)^{op}), \tilde{\mathcal{C}}') \times_{\mathrm{Fun}(\{\emptyset\}, \tilde{\mathcal{C}}')} \{X_{-1}\}$$

spanned by functors which are Čech nerves of atlases of X_{-1}. By construction, the definition of $\mathrm{Mon}_{\mathcal{P}\mathrm{f}}^{\mathcal{P}\mathrm{r}^{\mathrm{L}}_{\mathrm{st}}}(\mathcal{C}\mathrm{at}_\infty)$, and (P2), to prove Lemma 4.2.1, it suffices to show that for every morphism σ^0 of $\mathrm{Cov}(X_{-1})$, considered as a functor $\Delta^1 \times \mathrm{N}(\mathbf{\Delta}_+^{op}) \to \tilde{\mathcal{C}}'$, and every right Kan extension F of $\mathrm{Fun}(\mathcal{L}', \mathrm{G}) \circ {}_{\mathcal{C}'}\mathrm{EO}^*_\otimes \circ (\sigma^0 \mid \Delta^1 \times \mathrm{N}(\mathbf{\Delta}^{op}))^{op}$, $F \mid (\Delta^1 \times \{\emptyset\})^{op}$ is an equivalence in $\mathcal{P}\mathrm{r}^{\mathrm{L}}_{\mathrm{st}}$. We first prove a technical lemma.

Lemma 4.2.2. *Let $p\colon \mathcal{C} \to \mathcal{D}$ be a categorical fibration of ∞-categories. Let $c^{\bullet\bullet}\colon \mathrm{N}(\mathbf{\Delta}_+) \times \mathrm{N}(\mathbf{\Delta}_+) \to \mathcal{C}$ be an augmented bicosimplicial object of $\mathcal{C}$. For $n \geq -1$, let $c^{n\bullet} = c^{\bullet\bullet} \mid \{[n]\} \times \mathrm{N}(\mathbf{\Delta}_+)$ and $c^{\bullet n} = c^{\bullet\bullet} \mid \mathrm{N}(\mathbf{\Delta}_+) \times \{[n]\}$, respectively. Assume that*

(1) *$c^{\bullet\bullet}$ is a p-limit of $c^{\bullet\bullet} \mid \mathrm{N}(\mathbf{\Delta}_{++})$, where $\mathbf{\Delta}_{++} \subseteq \mathbf{\Delta}_+ \times \mathbf{\Delta}_+$ is the full subcategory spanned by all objects except the initial one.*
(2) *For every $n \geq 0$, $c^{n\bullet}$ is a p-limit of $c^{n\bullet} \mid \mathrm{N}(\mathbf{\Delta})$.*
(3) *For every $n \geq 0$, $c^{\bullet n}$ is a p-limit of $c^{\bullet n} \mid \mathrm{N}(\mathbf{\Delta})$.*

Then

(1) *$c^{-1\bullet}$ is a p-limit of $c^{-1\bullet} \mid \{\emptyset\} \times \mathrm{N}(\mathbf{\Delta})$.*
(2) *$c^{\bullet-1}$ is a p-limit of $c^{\bullet-1} \mid \mathrm{N}(\mathbf{\Delta}) \times \{\emptyset\}$.*
(3) *$c^{\bullet\bullet} \mid \mathrm{N}(\mathbf{\Delta}_+)_{\mathrm{diag}}$ is a p-limit of $c^{\bullet\bullet} \mid \mathrm{N}(\mathbf{\Delta})_{\mathrm{diag}}$, where $\mathrm{N}(\mathbf{\Delta}_+)_{\mathrm{diag}} \subseteq \mathrm{N}(\mathbf{\Delta}_+) \times \mathrm{N}(\mathbf{\Delta}_+)$ is the image of the diagonal inclusion $\mathrm{diag}\colon \mathrm{N}(\mathbf{\Delta}_+) \to \mathrm{N}(\mathbf{\Delta}_+) \times \mathrm{N}(\mathbf{\Delta}_+)$ and $\mathrm{N}(\mathbf{\Delta})_{\mathrm{diag}}$ is defined similarly.*

Proof. (1) We apply (the dual version of) [29, 4.3.2.8] to p and $\mathrm{N}(\mathbf{\Delta}_+ \times \mathbf{\Delta}) \subseteq \mathrm{N}(\mathbf{\Delta}_{++}) \subseteq \mathrm{N}(\mathbf{\Delta}_+ \times \mathbf{\Delta}_+)$. By (the dual version of) [29, 4.3.2.9] and assumption (2), $c^{\bullet\bullet} \mid \mathrm{N}(\mathbf{\Delta} \times \mathbf{\Delta}_+)$ is a p-right Kan extension of $c^{\bullet\bullet} \mid \mathrm{N}(\mathbf{\Delta} \times \mathbf{\Delta})$. It follows that $c^{\bullet\bullet} \mid \mathrm{N}(\mathbf{\Delta}_{++})$ is a p-right Kan extension of $c^{\bullet\bullet} \mid \mathrm{N}(\mathbf{\Delta}_+ \times \mathbf{\Delta})$. By assumption (1), $c^{\bullet\bullet}$ is a p-right Kan extension of $c^{\bullet\bullet} \mid \mathrm{N}(\mathbf{\Delta}_{++})$. Therefore, $c^{\bullet\bullet}$ is a p-right Kan extension of $c^{\bullet\bullet} \mid \mathrm{N}(\mathbf{\Delta}_+ \times \mathbf{\Delta})$. By [29, 4.3.2.9] again, $c^{-1\bullet}$ is a p-limit of $c^{-1\bullet} \mid \{\emptyset\} \times \mathrm{N}(\mathbf{\Delta})$.

(2) This follows from conclusion (1) by symmetry.

(3) We view $(\mathbf{\Delta} \times \mathbf{\Delta})^{\triangleleft}$ as a full subcategory of $\mathbf{\Delta}_+ \times \mathbf{\Delta}_+$ by sending the cone point to the initial object. By [29, 4.3.2.7], we find that $c^{\bullet\bullet} \mid (\mathbf{\Delta} \times \mathbf{\Delta})^{\triangleleft}$ is a p-limit diagram. By [29, 5.5.8.4], the simplicial set $\mathrm{N}(\mathbf{\Delta})^{op}$ is *sifted* [29, 5.5.8.1], *i.e.*, the diagonal map $\mathrm{N}(\mathbf{\Delta})^{op} \to \mathrm{N}(\mathbf{\Delta})^{op} \times \mathrm{N}(\mathbf{\Delta})^{op}$ is cofinal. Therefore, $c^{\bullet\bullet} \mid \mathrm{N}(\mathbf{\Delta}_+)_{\mathrm{diag}}$ is a p-limit of $c^{\bullet\bullet} \mid \mathrm{N}(\mathbf{\Delta})_{\mathrm{diag}}$.

□

Proof of Lemma 4.2.1. We show the assertion in the remark following the statement of Lemma 4.2.1. Let $\sigma\colon X^0_\bullet \to X^1_\bullet$ be a morphism of $\mathrm{Cov}(X_{-1})$. Let $X^2_\bullet$ be an object of $\mathrm{Cov}(X_{-1})$. Then we have a diagram

$$\begin{tikzcd}
 & X^0_\bullet \times X^2_\bullet \arrow[r, "\mathrm{pr}"] \arrow[dl, "\mathrm{pr}"'] \arrow[d, "\sigma \times X^2_\bullet"] & X^0_\bullet \arrow[d, "\sigma"] \\
X^2_\bullet & X^1_\bullet \times X^2_\bullet \arrow[l, "\mathrm{pr}"'] \arrow[r, "\mathrm{pr}"] & X^1_\bullet.
\end{tikzcd}$$

Here products are taken in $\mathrm{Cov}(X_{-1})$. Thus it suffices to show the assertion for the projection $X_\bullet \times X'_\bullet \to X'_\bullet$, where $X_\bullet$ and $X'_\bullet$ are objects of $\mathrm{Cov}(X_{-1})$.

Let $Y_{\bullet\bullet}\colon \mathrm{N}(\mathbf{\Delta}^{op}_+) \times \mathrm{N}(\mathbf{\Delta}^{op}_+) \to \tilde{\mathcal{C}}'$ be an augmented bisimplicial object of $\tilde{\mathcal{C}}'$ such that

- $Y_{-1\bullet} = X'$, $Y_{\bullet -1} = X$.
- $Y_{\bullet\bullet}$ is a right Kan extension of $Y_{-1\bullet} \cup Y_{\bullet -1}$.

Let $\delta\colon [1] \times \mathbf{\Delta}^{op}_+ \to \mathbf{\Delta}^{op}_+ \times \mathbf{\Delta}^{op}_+$ be the functor sending $(0,[n])$ (resp. $(1,[n])$) to $([n],[n])$ (resp. $([-1],[n])$). It suffices to show the assertion for $Y_{\bullet\bullet} \circ \mathrm{N}(\delta)$, which follows from Lemma 4.2.2 by taking p to be $\mathcal{P}\mathrm{r}^{\mathrm{L}}_{\mathrm{st}} \to *$ and $c^{\bullet\bullet}$ to be a right Kan extension of $\mathrm{Fun}(\mathcal{L}', \mathrm{G}) \circ {}_{\mathcal{C}'}\mathrm{EO}^*_{\otimes} \circ (Y_{\bullet\bullet} \mid \mathrm{N}(\mathbf{\Delta}^{op}_{++}))^{op}$. Assumptions (2) and (3) of Lemma 4.2.2 are satisfied thanks to (P4). □

Since res_1 is a trivial fibration [29, 4.3.2.15], $\mathcal{N}(\sigma_n)$ is weakly contractible. By Lemma 4.2.1, we can apply Lemma 1.2.2 to $K = \delta^*_{2,\{2\}}\mathrm{Fun}(\Delta^1, \tilde{\mathcal{C}}')^{\mathrm{cart}}_{\mathcal{R}^0, \tilde{\mathcal{C}}'^{\to}_1}$, $K' = \delta^2_{2,\{2\}}\mathrm{Fun}(\Delta^1, \mathcal{C}')^{\mathrm{cart}}_{(\mathcal{F}' \cap \mathcal{C}'_1)^0, \mathcal{C}'^{\to}_1}$, the inclusion $g\colon K' \to K$ and the section ν given by ${}_{\mathcal{C}'}\mathrm{EO}$. This extends ${}_{\mathcal{C}'}\mathrm{EO}$ to a map

$$\tag{4.8} {}^{\mathcal{R}}_{\tilde{\mathcal{C}}'}\mathrm{EO}\colon \delta^*_{2,\{2\}}\mathrm{Fun}(\Delta^1, \tilde{\mathcal{C}}')^{\mathrm{cart}}_{\mathcal{R}^0, \tilde{\mathcal{C}}'^{\to}_1} \to \mathrm{Fun}(\mathcal{L}', \mathrm{Mon}^{\mathcal{P}\mathrm{r}^{\mathrm{L}}_{\mathrm{st}}}_{\mathcal{P}\mathrm{f}}(\mathcal{C}\mathrm{at}_\infty)).$$

Step 2. Now we are going to extend ${}^{\mathcal{R}}_{\tilde{\mathcal{C}}'}\mathrm{EO}$ to $\delta^*_{2,\{2\}}\mathrm{Fun}(\Delta^1, \tilde{\mathcal{C}}')^{\mathrm{cart}}_{(\tilde{\mathcal{F}} \cap \tilde{\mathcal{C}}'_1)^0, \tilde{\mathcal{C}}'^{\to}_1}$. An n-cell of $\delta^*_{2,\{2\}}\mathrm{Fun}(\Delta^1, \tilde{\mathcal{C}}')^{\mathrm{cart}}_{(\tilde{\mathcal{F}} \cap \tilde{\mathcal{C}}'_1)^0, \tilde{\mathcal{C}}'^{\to}_1}$ is given by a functor $\varsigma_n\colon \Delta^n \times (\Delta^n)^{op} \to \mathrm{Fun}(\Delta^1, \tilde{\mathcal{C}}')$. We define $\mathrm{Kov}(\varsigma_n)$ to be the full subcategory of

$$\mathrm{Fun}(\Delta^n \times (\Delta^n)^{op} \times \mathrm{N}(\mathbf{\Delta}^{op}_+), \mathrm{Fun}(\Delta^1, \tilde{\mathcal{C}}')) \times_{\mathrm{Fun}(\Delta^n \times (\Delta^n)^{op} \times \{\emptyset\}, \mathrm{Fun}(\Delta^1, \tilde{\mathcal{C}}'))} \{\varsigma\}$$

spanned by functors $\varsigma^0_n\colon \Delta^n \times (\Delta^n)^{op} \times \mathrm{N}(\mathbf{\Delta}^{op}_+) \to \mathrm{Fun}(\Delta^1, \tilde{\mathcal{C}}')$ such that

- for every object (i,j) of $\Delta^n \times (\Delta^n)^{op}$, the restriction $\varsigma^0_n \mid \Delta^{\{(i,j)\}} \times \mathrm{N}((\mathbf{\Delta}^{\leq 0}_+)^{op})$ is given by the square

$$\tag{4.9} \begin{tikzcd}
Y^{i,j}_0 \arrow[r] \arrow[d, "g^{i,j}"'] & X^{i,j}_0 \arrow[d, "f^{i,j}"] \\
Y^{i,j}_{-1} \arrow[r] & X^{i,j}_{-1}
\end{tikzcd}$$

where $f^{i,j}$ and $g^{i,j}$ are morphisms in $\tilde{\mathcal{E}}_{\mathrm{s}} \cap \tilde{\mathcal{E}}' \cap \mathcal{R}$;
- ς^0_n is a right Kan extension of $\varsigma^0_n \mid \Delta^n \times (\Delta^{\{0\}})^{op} \times \mathrm{N}((\mathbf{\Delta}^{\leq 0}_+)^{op}) \cup \Delta^n \times (\Delta^n)^{op} \times \{\emptyset\}$;

- the restriction $\varsigma_n^0 \mid \Delta^n \times (\Delta^n)^{op} \times \{[0]\} \times \Delta^1$ corresponds to an n-cell of $\delta^*_{2,\{2\}}\mathrm{Fun}(\Delta^1, \tilde{\mathcal{C}}')^{\mathrm{cart}}_{\mathcal{R}^0, \tilde{\mathcal{C}}_1'^{\to}}$.

In particular, objects ς_n^0 of $\mathrm{Kov}(\varsigma_n)$ satisfy

- for every object (i,j) of $\Delta^n \times (\Delta^n)^{op}$, $\varsigma_n^0 \mid \Delta^{\{(i,j)\}} \times \mathrm{N}(\mathbf{\Delta}_+^{op})$ is a Čech nerve of (4.9).

Similarly to $\mathrm{Cov}(\sigma)$, the ∞-category $\mathrm{Kov}(\varsigma_n)$ is nonempty and admits product of two objects. Therefore, by Lemma 1.1.1, $\mathrm{Kov}(\varsigma_n)$ is a weakly contractible Kan complex.

The restriction functor

$$\mathrm{Kov}(\varsigma_n) \to \mathrm{Fun}(\mathrm{N}(\mathbf{\Delta}^{op}) \times \Delta^n \times (\Delta^n)^{op}, \mathrm{Fun}(\Delta^1, \tilde{\mathcal{C}}'))$$

induces a map

$$\mathrm{Kov}(\varsigma_n) \to \mathrm{Fun}(\mathrm{N}(\mathbf{\Delta}^{op}), \mathrm{Fun}(\Delta^n, \delta^*_{2,\{2\}}\mathrm{Fun}(\Delta^1, \tilde{\mathcal{C}}')^{\mathrm{cart}}_{\mathcal{R}^0, \tilde{\mathcal{C}}_1'^{\to}})).$$

Composing with the map ${}^{\mathcal{R}}_{\tilde{\mathcal{C}}'}\mathrm{EO}$, we obtain a map

$$\phi(\varsigma_n) \colon \mathrm{Kov}(\varsigma_n) \to \mathrm{Fun}(\mathrm{N}(\mathbf{\Delta}^{op}), \mathrm{Fun}(\Delta^n, \mathrm{Fun}(\mathcal{L}', \mathrm{Mon}^{\mathcal{P}\mathrm{r}^{\mathrm{L}}_{\mathrm{st}}}_{\mathcal{P}\mathrm{f}}(\mathcal{C}\mathrm{at}_\infty)))).$$

Let $\mathcal{K}' \subseteq \mathrm{Fun}(\mathrm{N}(\mathbf{\Delta}_+^{op}), \mathrm{Fun}(\Delta^n, \mathrm{Fun}(\mathcal{L}', \mathrm{Mon}^{\mathcal{P}\mathrm{r}^{\mathrm{L}}_{\mathrm{st}}}_{\mathcal{P}\mathrm{f}}(\mathcal{C}\mathrm{at}_\infty))))$ be the full subcategory spanned by those functors $F \colon \mathrm{N}(\mathbf{\Delta}_+^{op}) \to \mathrm{Fun}(\Delta^n, \mathrm{Fun}(\mathcal{L}', \mathrm{Mon}^{\mathcal{P}\mathrm{r}^{\mathrm{L}}_{\mathrm{st}}}_{\mathcal{P}\mathrm{f}}(\mathcal{C}\mathrm{at}_\infty)))$ which are left Kan extensions of $F \mid \mathrm{N}(\mathbf{\Delta}^{op})$. Consider the following diagram

$$\begin{tikzcd}
\mathcal{N}(\varsigma_n) \arrow[rr] \arrow[d, "\mathrm{res}_1^*\phi(\varsigma_n)"'] & & \mathrm{Kov}(\varsigma_n) \arrow[d, "\phi(\varsigma_n)"] \\
\mathcal{K}' \arrow[rr, "\mathrm{res}_1"] \arrow[d, "\mathrm{res}_2"'] & & \mathrm{Fun}(\mathrm{N}(\mathbf{\Delta}^{op}), \mathrm{Fun}(\Delta^n, \mathrm{Fun}(\mathcal{L}', \mathrm{Mon}^{\mathcal{P}\mathrm{r}^{\mathrm{L}}_{\mathrm{st}}}_{\mathcal{P}\mathrm{f}}(\mathcal{C}\mathrm{at}_\infty)))) \\
\mathrm{Fun}(\Delta^n, \mathrm{Fun}(\mathcal{L}', \mathrm{Mon}^{\mathcal{P}\mathrm{r}^{\mathrm{L}}_{\mathrm{st}}}_{\mathcal{P}\mathrm{f}}(\mathcal{C}\mathrm{at}_\infty))), & &
\end{tikzcd}$$

where the upper square is Cartesian, and let $\Phi(\varsigma_n) = \mathrm{res}_2 \circ \mathrm{res}_1^*\phi(\varsigma_n)$. It is easy to see that the above process is functorial so that the collection of $\Phi(\varsigma_n)$ defines a morphism Φ in the category $(\mathcal{S}\mathrm{et}_\Delta)^{(\mathbf{\Delta}_{\delta^*_{2,\{2\}}\mathrm{Fun}(\Delta^1, \tilde{\mathcal{C}}')^{\mathrm{cart}}_{(0,\to)}})^{op}}$.

Lemma 4.2.3. *The map $\Phi(\varsigma_n)$ takes values in* $\mathrm{Map}^\sharp((\Delta^n)^\flat, \mathrm{Fun}(\mathcal{L}', \mathrm{Mon}^{\mathcal{P}\mathrm{r}^{\mathrm{L}}_{\mathrm{st}}}_{\mathcal{P}\mathrm{f}}(\mathcal{C}\mathrm{at}_\infty))^\natural)$.

Proof. Let ${}^{\mathcal{R}}_{\tilde{\mathcal{C}}'}\mathrm{EO}^*_!$ be the composition

$$\delta^*_{2,\{2\}}\tilde{\mathcal{C}}'^{\mathrm{cart}}_{\mathcal{R}, \tilde{\mathcal{C}}_1'} \to \delta^*_{2,\{2\}}\mathrm{Fun}(\Delta^1, \tilde{\mathcal{C}}')^{\mathrm{cart}}_{\mathcal{R}^0, \tilde{\mathcal{C}}_1'^{\to}} \xrightarrow{{}^{\mathcal{R}}_{\tilde{\mathcal{C}}'}\mathrm{EO}} \mathrm{Fun}(\mathcal{L}', \mathrm{Mon}^{\mathcal{P}\mathrm{r}^{\mathrm{L}}_{\mathrm{st}}}_{\mathcal{P}\mathrm{f}}(\mathcal{C}\mathrm{at}_\infty)) \xrightarrow{\mathrm{Fun}(\mathcal{L}', \mathrm{G}_{(\langle 1\rangle, \{1\})})} \mathrm{Fun}(\mathcal{L}', \mathcal{P}\mathrm{r}^{\mathrm{L}}_{\mathrm{st}}),$$

where the first map is induced by Σ appearing in Input I. Let $X_\bullet \colon \mathrm{N}(\mathbf{\Delta}_+^{op}) \to \tilde{\mathcal{C}}'$ be an augmented simplicial object such that $X_\bullet$ is the Čech nerve of $f \colon X_0 \to X_{-1}$ and f is in $\tilde{\mathcal{E}}_{\mathrm{s}} \cap \tilde{\mathcal{E}}' \cap \mathcal{R}$. By the construction of $\Phi(\varsigma_n)$, it suffices to show that $R \circ X_\bullet$ is a left Kan extension of $R \circ X_\bullet \mid \mathrm{N}(\mathbf{\Delta}^{op})$. Here $R = {}^{\mathcal{R}}_{\tilde{\mathcal{C}}'}\mathrm{EO}^*_! \mid \tilde{\mathcal{C}}'_{\mathcal{R}}$ is the restriction to direction 1. Choose an object $X'_\bullet$ of $\mathrm{Cov}(X_{-1})$ and form a bisimplicial object $Y_{\bullet\bullet} \colon \mathrm{N}(\mathbf{\Delta}_+^{op} \times \mathbf{\Delta}_+^{op}) \to \tilde{\mathcal{C}}'$ as in the proof of Lemma 4.2.1. Applying ${}^{\mathcal{R}}_{\tilde{\mathcal{C}}'}\mathrm{EO}$ to $Y_{\bullet\bullet}$, we obtain a diagram $\chi_{\bullet\bullet} \colon \mathrm{N}(\mathbf{\Delta}_+^{op}) \times \mathrm{N}(\mathbf{\Delta}_+) \to \mathcal{P}\mathrm{r}^{\mathrm{L}}_{\mathrm{st}}$. By the construction of ${}^{\mathcal{R}}_{\tilde{\mathcal{C}}'}\mathrm{EO}$, $\chi_{n\bullet}$ is a limit diagram for $n \geq -1$. By (P4), $\chi_{\bullet n}$ is a colimit diagram for $n \geq 0$. Therefore, by (P5) (2) and [30, 6.2.3.19] applied to the restriction $\chi_{\bullet\bullet} \mid \mathrm{N}(\mathbf{\Delta}_{s,+}^{op}) \times \mathrm{N}(\mathbf{\Delta}_{s,+})$, $R \circ X_\bullet = \chi_{\bullet -1}$ is a colimit diagram. In the last sentence, we use [29, 6.3.5.7] twice. □

Since res_1 is a trivial fibration, $\mathcal{N}(\varsigma_n)$ is weakly contractible. By the previous lemma, we can apply Lemma 1.2.2 to $K = \delta^*_{2,\{2\}}\mathrm{Fun}(\Delta^1, \tilde{\mathcal{C}}')^{\mathrm{cart}}_{(\tilde{\mathcal{F}} \cap \tilde{\mathcal{C}}_1')^0, \tilde{\mathcal{C}}_1'^{\to}}$, $K' = \delta^*_{2,\{2\}}\mathrm{Fun}(\Delta^1, \tilde{\mathcal{C}}')^{\mathrm{cart}}_{\mathcal{R}^0, \tilde{\mathcal{C}}_1'^{\to}}$, the inclusion $g \colon K' \to K$ and the section ν given by ${}^{\mathcal{R}}_{\tilde{\mathcal{C}}'}\mathrm{EO}$. This extends ${}^{\mathcal{R}}_{\tilde{\mathcal{C}}'}\mathrm{EO}$ to a map

$$_{\tilde{\mathcal{C}}'}\mathrm{EO} \colon \delta^*_{2,\{2\}}\mathrm{Fun}(\Delta^1, \tilde{\mathcal{C}}')^{\mathrm{cart}}_{(\tilde{\mathcal{F}} \cap \tilde{\mathcal{C}}_1')^0, \tilde{\mathcal{C}}_1'^{\to}} \to \mathrm{Fun}(\mathcal{L}', \mathrm{Mon}^{\mathcal{P}\mathrm{r}^{\mathrm{L}}_{\mathrm{st}}}_{\mathcal{P}\mathrm{f}}(\mathcal{C}\mathrm{at}_\infty)). \tag{4.10}$$

Proof of 4.1.1. The proof is similar to Step 1 above. Consider a diagram

$$\begin{tikzcd}
\partial\Delta^n \arrow[r, "G"] \arrow[d, hook] & \mathrm{Fun}^{\tilde{\mathcal{E}}}(\tilde{\mathcal{C}}^{op}, \mathcal{D}) \arrow[d] \\
\Delta^n \arrow[r, "F"'] \arrow[ur, dashed] & \mathrm{Fun}^{\mathcal{E}}(\mathcal{C}^{op}, \mathcal{D}).
\end{tikzcd}$$

Let $\sigma\colon (\Delta^m)^{op} \to \mathcal{C}$ be an m-cell of $\tilde{\mathcal{C}}^{op}$. We denote by $\mathrm{Cov}(\sigma)$ the full subcategory of

$$\mathrm{Fun}((\Delta^m)^{op} \times \mathrm{N}(\mathbf{\Delta}_+^{op}), \tilde{\mathcal{C}}) \times_{\mathrm{Fun}((\Delta^m)^{op}\times\{[-1]\}, \tilde{\mathcal{C}})} \{\sigma\}$$

spanned by Čech nerves $\sigma^0\colon (\Delta^m)^{op} \times \mathrm{N}(\mathbf{\Delta}_+^{op}) \to \tilde{\mathcal{C}}$ such that $\sigma^0 \mid (\Delta^m)^{op} \times \mathrm{N}(\mathbf{\Delta}^{op})$ factors through $\mathcal{C}$, and that $\sigma^0 \mid \Delta^{\{j\}} \times \mathrm{N}((\mathbf{\Delta}_+^{\le 0})^{op})$ belongs to $\tilde{\mathcal{E}}$ and is representable in $\mathcal{C}$ for all $0 \le j \le m$. Since $\mathrm{Cov}(\sigma)$ admits product of two objects, it is a contractible Kan complex by Lemma 1.1.1.

Let $\mathcal{K} \subseteq \mathrm{Fun}(\mathrm{N}(\mathbf{\Delta}_+), \mathrm{Fun}(\Delta^m, \mathcal{D}))$ be the full subcategories spanned by augmented cosimplicial objects $X_\bullet^+$ that are right Kan extensions of $X_\bullet^+ \mid \mathrm{N}(\mathbf{\Delta})$. By [29, 4.3.2.15], the restriction map $\mathcal{K} \to \mathrm{Fun}(\mathrm{N}(\mathbf{\Delta}), \mathrm{Fun}(\Delta^m, \mathcal{D}))$ is a trivial fibration. We have a diagram

$$\begin{tikzcd}
\mathrm{Cov}(\sigma)^{op} \arrow[dr, "\phi"] \arrow[drr, "\alpha"] \arrow[ddr, "\beta"'] & & \\
& \mathcal{K}' \arrow[r] \arrow[d] & \mathrm{Fun}(\Delta^n, \mathrm{Fun}(\mathrm{N}(\mathbf{\Delta}) \times \Delta^m, \mathcal{D})) \arrow[d] \\
& \mathrm{Fun}(\partial\Delta^n, \mathcal{K}) \arrow[r] & \mathrm{Fun}(\partial\Delta^n, \mathrm{Fun}(\mathrm{N}(\mathbf{\Delta}) \times \Delta^m, \mathcal{D}))
\end{tikzcd}$$

where the square is Cartesian, α is induced by F, and β is induced by G. Consider the diagram

$$\begin{tikzcd}
& \mathcal{N}(\sigma) \arrow[r] \arrow[d, "\mathrm{res}_1^*\phi"'] & \mathrm{Cov}(\sigma)^{op} \arrow[d, "\phi"] \\
\mathrm{Fun}(\Delta^n, \mathrm{Fun}(\Delta^m, \mathcal{D})) & \mathrm{Fun}(\Delta^n, \mathcal{K}) \arrow[l, "\mathrm{res}_2"'] \arrow[r, "\mathrm{res}_1"] & \mathcal{K}',
\end{tikzcd}$$

where the square is Cartesian and res_2 is the restriction to $\{[-1]\}$. Since res_1 is a trivial fibration, $\mathcal{N}(\sigma)$ is a contractible Kan complex.

Let $\Phi(\sigma) = \mathrm{res}_2 \circ \mathrm{res}_1^*\phi$. This is functorial in σ in the sense that it defines a morphism Φ in the category $(\mathcal{S}\mathrm{et}_\Delta)^{(\mathbf{\Delta}_{\tilde{\mathcal{C}}^{op}})^{op}}$. Moreover, $\Phi(\sigma)$ takes values in $\mathrm{Map}^\sharp((\Delta^m)^\flat, \mathrm{Fun}(\Delta^n, \mathcal{D})^\natural)$. In fact, this is trivial for $n > 0$ and the proof of Lemma 4.2.1 can be easily adapted to treat the case $n = 0$. Applying Lemma 1.2.1 to Φ and $a = G$, we obtain a lifting $\tilde{F}\colon \Delta^n \to \mathrm{Fun}(\tilde{\mathcal{C}}^{op}, \mathcal{D})$ of F extending G.

It remains to show that $\tilde{F}$ factorizes through $\mathrm{Fun}^{\tilde{\mathcal{E}}}(\tilde{\mathcal{C}}^{op}, \mathcal{D})$. This is trivial for $n > 0$. For $n = 0$, we need to show that every morphism $f\colon Y \to X$ in $\tilde{\mathcal{E}}$ is of $\tilde{F}$-descent, where we regard $\tilde{F}$ as a functor $\mathcal{C}^{op} \to \mathcal{D}$. Let $u\colon X' \to X$ be a morphism in $\tilde{\mathcal{E}}$ with X' in $\mathcal{C}$, and v be the composite morphism $Y' \xrightarrow{w} Y \times_X X' \to Y$ of the pullback of u and a morphism w in $\mathcal{E}$ with Y' in $\mathcal{C}$. This provides a diagram

$$\begin{tikzcd}
Y' \arrow[r, "f'"] \arrow[d, "v"'] & X' \arrow[d, "u"] \\
Y \arrow[r, "f"] & X
\end{tikzcd}$$

where u and v are in $\tilde{\mathcal{E}}$ and f' is in $\mathcal{E}$. Then f' and u are of $\tilde{F}$-descent by construction. It follows that f is of F-descent by Lemma 3.1.2 (3), (4). □

Applying Proposition 4.1.1 to $\tilde{\mathcal{E}} = \tilde{\mathcal{E}}_{\mathrm{s}} \cap \tilde{\mathcal{E}}''$, we obtain ${}_{\tilde{\mathcal{C}}}\mathrm{EO}^*_\otimes$ satisfying the first requirement of (P4).

4.3. **Properties.** We construct Output II and prove that Output I and Output II satisfy all required properties.

Lemma 4.3.1 (P1)**.** *The map* ${}_{\tilde{\mathcal{C}}}\mathrm{EO}^*_{\otimes}$ *sends small coproducts to products.*

Proof. Since $\tilde{\mathcal{C}}'$ is geometric (Definition 4.1.3), small coproducts commute with pullbacks. Therefore, forming Čech nerves commutes with the disjoint union. Then the lemma follows from the construction of ${}_{\tilde{\mathcal{C}}}\mathrm{EO}^*_{\otimes}$ and the property (P1) for ${}_{\mathcal{C}}\mathrm{EO}^*_{\otimes}$. □

Lemma 4.3.2 (P2)**.** *The map* $\mathrm{Fun}(\mathcal{L}', \mathrm{pf}) \circ \mathrm{Fun}((\Delta^1)^{op}, ({}_{\tilde{\mathcal{C}}}\mathrm{EO}^*_{\otimes} \mid \tilde{\mathcal{C}}'^{op}))$ *is equivalent to* ${}_{\tilde{\mathcal{C}}'}\mathrm{EO}^*_{\mathrm{pf}}$*, which is defined by a formula similar to* (4.4)*.*

Proof. Using the arguments at the end of §4.2, one shows that both maps belong to $\mathrm{Fun}^{\tilde{\mathcal{E}}_{\mathrm{s}} \cap \tilde{\mathcal{E}}'' \cap \tilde{\mathcal{C}}'_1}(\tilde{\mathcal{C}}^{op}, \mathrm{Mon}^{\mathcal{P}\mathrm{r}^{\mathrm{L}}_{\mathrm{st}}}_{\mathcal{P}\mathrm{f}}(\mathcal{C}\mathrm{at}_\infty))$ (see also the proof of (P4)). Moreover, their restrictions to $\mathcal{C}^{op}$ are equivalent, by (P2) for the Input. It then suffices to apply Proposition 4.1.1. □

Lemma 4.3.3 (P3)**.** *The functor* f^* *is conservative for every* $f\colon Y \to X$ *in* $\tilde{\mathcal{E}}_{\mathrm{s}}$*.*

Proof. We may put f into the following diagram

$$\begin{tikzcd} Y' \arrow[r, "f'"] \arrow[d, "v"'] & X' \arrow[d, "u"] \\ Y \arrow[r, "f"] & X \end{tikzcd}$$

where u is an atlas, Y is in $\mathcal{C}$ and g is in $\mathcal{E}_{\mathrm{s}}$. Then we only need to show that $v^* \circ f^*$, which is equivalent to $f'^* \circ u^*$, is conservative. By [30, 6.2.4.2 (3)], u^* is conservative, and f'^* is also conservative by the original (P3). Therefore, f^* is conservative. □

Proposition 4.3.4 (P4)**.** *Let* $f\colon Y \to X$ *be a morphism of* $\tilde{\mathcal{C}}$*. Then*

(1) f *is of universal* ${}_{\tilde{\mathcal{C}}}\mathrm{EO}^*_{\otimes}$*-descent if* f *is in* $\tilde{\mathcal{E}}_{\mathrm{s}} \cap \tilde{\mathcal{E}}''$*.*
(2) f *is of universal* ${}_{\tilde{\mathcal{C}}'}\mathrm{EO}_!$*-codescent if* f *is in* $\tilde{\mathcal{E}}_{\mathrm{s}} \cap \tilde{\mathcal{E}}'' \cap \tilde{\mathcal{C}}'_1$*.*

Although the first part has already been proved in Proposition 4.1.1, we will write the proof for both since they are same.

Proof. By construction, the assertions are true if f is an atlas. Moreover, by the original (P4), the assertions are also true if f is a morphism of $\mathcal{C}$. In the general case, consider a diagram

$$\begin{tikzcd} Y' \arrow[r, "f'"] \arrow[d, "v"'] & X' \arrow[d, "u"] \\ Y \arrow[r, "f"] & X \end{tikzcd}$$

where u is an atlas and f' is in $\mathcal{E}_{\mathrm{s}} \cap \mathcal{E}''$. For example, we can take v to be an atlas of $Y \times_X X'$. The proposition then follows from Lemma 3.1.2 (3), (4) and dual statements. □

We will only check (P5), and (P5$^{\mathrm{bis}}$) follows in the same way.

Proposition 4.3.5 (P5)**.** *Let*

$$\begin{tikzcd} W \arrow[r, "g"] \arrow[d, "q"'] & Z \arrow[d, "p"] \\ Y \arrow[r, "f"] & X \end{tikzcd}$$

be a Cartesian diagram of $\tilde{\mathcal{C}}'$ *with* f *in* $\tilde{\mathcal{E}}'$*, and* λ *be an object of* $\tilde{\mathcal{L}}'$*. Then*

(1) *The square*

$$\begin{tikzcd} \mathcal{D}(Z,\lambda) \arrow[d, "g^*"'] & \mathcal{D}(X,\lambda) \arrow[l, "p^*"'] \arrow[d, "f^*"] \\ \mathcal{D}(W,\lambda) & \mathcal{D}(Y,\lambda) \arrow[l, "q^*"'] \end{tikzcd} \tag{4.11}$$

has a right adjoint that is a square of $\mathcal{P}\mathrm{r}^{\mathrm{R}}_{\mathrm{st}}$.

(2) *If p is also in $\tilde{\mathcal{E}}'$, the square*

$$\begin{tikzcd} \mathcal{D}(X,\lambda) \arrow[d, "p^*"'] & \mathcal{D}(Y,\lambda) \arrow[l, "f_!"'] \arrow[d, "q^*"] \\ \mathcal{D}(Z,\lambda) & \mathcal{D}(W,\lambda) \arrow[l, "g_!"'] \end{tikzcd} \tag{4.12}$$

is right adjointable.

We first prove a technical lemma.

Lemma 4.3.6. *Let K be a simplicial set and $p\colon K \to \mathrm{Fun}(\Delta^1 \times \Delta^1, \mathcal{C}\mathrm{at}_\infty)$ be a diagram of squares of ∞-categories. We view p as a functor $K \times \Delta^1 \times \Delta^1 \to \mathcal{C}\mathrm{at}_\infty$. If for every 1-cell $\sigma\colon \Delta^1 \to K \times \Delta^1$, the induced square $p \circ (\sigma \times \mathrm{id}_{\Delta^1})\colon \Delta^1 \times \Delta^1 \to \mathcal{C}\mathrm{at}_\infty$ is right adjointable (resp. left adjointable), then the limit square $\varprojlim(p)$ is right adjointable (resp. left adjointable).*

Proof. Let us prove the right adjointable case, the proof of the other case being essentially the same. The assumption allows us to view p as a functor $p'\colon K \to \mathrm{Fun}(\Delta^1, \mathrm{Fun}^{\mathrm{RAd}}(\Delta^1, \mathcal{C}\mathrm{at}_\infty))$ [30, 6.2.3.16]. By [30, 6.2.3.18] and (the dual version of) [29, 5.1.2.3], the ∞-category $\mathrm{Fun}(\Delta^1, \mathrm{Fun}^{\mathrm{RAd}}(\Delta^1, \mathcal{C}\mathrm{at}_\infty))$ admits all limits and these limits are preserved by the inclusion

$$\mathrm{Fun}(\Delta^1, \mathrm{Fun}^{\mathrm{RAd}}(\Delta^1, \mathcal{C}\mathrm{at}_\infty)) \subseteq \mathrm{Fun}(\Delta^1, \mathrm{Fun}(\Delta^1, \mathcal{C}\mathrm{at}_\infty)).$$

Therefore, the limit square $\varprojlim(p)$ is equivalent to $\varprojlim(p')$ which is right adjointable. □

Proof of Proposition 4.3.5. (1) It is clear from the construction and the original (P5) (1) that both f^* and g^* admit left adjoints. Therefore, we only need to show that (4.11) is right adjointable. By Lemma 4.3.6, we may assume that f is in $\mathcal{E}'$. Then it reduces to show that the transpose of (4.11) is left adjointable, which allows us to assume that p is a morphism in $\mathcal{C}'$, again by Lemma 4.3.6. Then it follows from the original (P5) (1).

(2) By Lemma 4.3.6, we may assume that p is in $\mathcal{E}'$. Then p^* and q^* admit left adjoints. Therefore, we only need to prove that the transpose of (4.12) is left adjointable, which allows us to assume that f is also in $\mathcal{E}'$, again by Lemma 4.3.6. Then it follows from the original (P5) (2).

□

Next we define the t-structure. Let X be an object of $\tilde{\mathcal{C}}$ and λ be an object of $\mathcal{L}$. For an atlas $f\colon X_0 \to X$, we denote by $\mathcal{D}^{\leq 0}_f(X,\lambda) \subseteq \mathcal{D}(X,\lambda)$ (resp. $\mathcal{D}^{\geq 0}_f(X,\lambda) \subseteq \mathcal{D}(X,\lambda)$) the full subcategory spanned by complexes $\mathscr{K}$ such that $f^*\mathscr{K}$ is in $\mathcal{D}^{\leq 0}(X_0,\lambda)$ (resp. $\mathcal{D}^{\geq 0}(X_0,\lambda)$).

Lemma 4.3.7. *We have*

(1) *The pair of subcategories $(\mathcal{D}^{\leq 0}_f(X,\lambda), \mathcal{D}^{\geq 0}_f(X,\lambda))$ determine a t-structure on $\mathcal{D}(X,\lambda)$.*

(2) *The pair of subcategories $(\mathcal{D}^{\leq 0}_f(X,\lambda), \mathcal{D}^{\geq 0}_f(X,\lambda))$ do not depend on the choice of f.*

In what follows, we will write $(\mathcal{D}^{\leq 0}(X,\lambda), \mathcal{D}^{\geq 0}(X,\lambda)) = (\mathcal{D}^{\leq 0}_f(X,\lambda), \mathcal{D}^{\geq 0}_f(X,\lambda))$ for an atlas f. Moreover, if X is an object of $\mathcal{C}$, then the new t-structure coincides with the old one since $\mathrm{id}_X\colon X \to X$ is an atlas.

Proof. (1) Let $f_\bullet\colon X_\bullet \to X$ be a Čech nerve of $f_0 = f$. We need to check the axioms of [30, 1.2.1.1]. To check axiom (1), let $\mathscr{K}$ be an object of $\mathcal{D}_f^{\le 0}(X,\lambda)$ and $\mathscr{L}$ be an object of $\mathcal{D}_f^{\ge 1}(X,\lambda)$. By (P6) for the input and Proposition 4.3.4 (1), we have equivalences

$$\mathrm{Hom}(\mathscr{K},\mathscr{L}) \simeq \mathrm{Hom}(\mathscr{K}, \varprojlim_{n\in\mathbf{\Delta}} f_{n*}f_n^*\mathscr{L}) \simeq \varprojlim_{n\in\mathbf{\Delta}} \mathrm{Hom}(f_n^*\mathscr{K}, f_n^*\mathscr{L}) = 0.$$

Axiom (2) is trivial. By (P6) for the input, we have a cosimplicial diagram $p\colon \mathrm{N}(\mathbf{\Delta}) \to \mathrm{Fun}(\Delta^1, \mathcal{C}\mathrm{at}_\infty)$ sending $[n]$ to the functor $\mathcal{D}(X_n,\lambda) \to \mathrm{Fun}(\Delta^1\times\Delta^1, \mathcal{D}(X_n,\lambda))$ that corresponds to the following Cartesian diagram of functors:

$$\begin{tikzcd} \tau_n^{\le 0} \arrow[r] \arrow[d] & \mathrm{id}_{X_n} \arrow[d] \\ 0 \arrow[r] & \tau_n^{\ge 1}, \end{tikzcd}$$

where $\tau_n^{\le 0}$ and $\tau_n^{\ge 1}$ (resp. id_{X_n}) are the truncation functors (resp. is the identity functor) of $\mathcal{D}(X_n,\lambda)$. Axiom (3) follows from the fact that $\varprojlim(p)$ provides a similar Cartesian diagram of endofunctors of $\mathcal{D}(X,\lambda)$.

(2) By (1), it suffices to show that for every atlas $f'\colon X_0' \to X$, $\mathcal{D}_f^{\le 0}(X,\lambda) = \mathcal{D}_{f'}^{\le 0}(X,\lambda)$. Let $\mathscr{K}$ be an object of $\mathcal{D}_f^{\le 0}(X,\lambda)$ and form a Cartesian diagram

$$\begin{tikzcd} Y \arrow[r, "g"] \arrow[d, "g'"'] & X_0' \arrow[d, "f'"] \\ X_0 \arrow[r, "f"] & X. \end{tikzcd}$$

By (P6) for the input, g^* and g'^* are t-exact, so that

$$g^*\tau^{\ge 1}f'^*\mathscr{K} \simeq \tau^{\ge 1}g^*f'^*\mathscr{K} \simeq \tau^{\ge 1}g'^*f^*\mathscr{K} \simeq g'^*\tau^{\ge 1}f^*\mathscr{K} = 0.$$

Since g^* is conservative by (P3) for the input, $\tau^{\ge 1}f'^*\mathscr{K} = 0$. In other words, $f'^*\mathscr{K}$ belongs to $\mathcal{D}^{\le 0}(X_0',\lambda)$. Therefore, $\mathcal{D}_f^{\le 0}(X,\lambda) \subseteq \mathcal{D}_{f'}^{\le 0}(X,\lambda)$. By symmetry, $\mathcal{D}_f^{\le 0}(X,\lambda) \supseteq \mathcal{D}_{f'}^{\le 0}(X,\lambda)$. It follows that $\mathcal{D}_f^{\le 0}(X,\lambda) = \mathcal{D}_{f'}^{\le 0}(X,\lambda)$. □

Parts (1) and (2) of (P6) are obvious from the constructions.

Lemma 4.3.8 (P6 (3))**.** *For every morphism $f\colon Y \to X$ of $\tilde{\mathcal{C}}$, f^* is t-exact with respect to the above t-structure.*

Proof. Put $f\colon \mathcal{Y} \to \mathcal{X}$ into a diagram

$$\begin{tikzcd} Y' \arrow[r, "f'"] \arrow[d, "v"'] & X' \arrow[d, "u"] \\ Y \arrow[r, "f"] & X \end{tikzcd}$$

where u, v are atlases. Then the assertion follows from the definitions and the fact that f'^* is t-exact. □

Finally we construct the trace maps. We will construct the trace maps for $\tilde{\mathcal{E}}_{\mathrm{t}}$ and check (P7). Construction of the trace maps for $\tilde{\mathcal{E}}'$ and verification of (P7$^{\mathrm{bis}}$) are similar and in fact easier.

Lemma 4.3.9. *There exists a unique way to define the trace map*

$$\mathrm{Tr}_f\colon \tau^{\ge 0}f_!\lambda_Y\langle d\rangle \to \lambda_X,$$

for morphisms $f\colon Y \to X$ in $\mathcal{R}\cap\tilde{\mathcal{E}}_{\mathrm{t}}\cap\tilde{\mathcal{C}}_1''$ and integers $d \ge \dim^+(f)$, satisfying (P7) (1) and extending the input. In particular, for such a morphism f, $f_!\lambda_Y\langle d\rangle$ is in $\mathcal{D}^{\le 0}(X,\lambda)$.

Proof. Let

$$\begin{tikzcd} Y_0 \arrow[r, "f_0"] \arrow[d, "y_0"'] & X_0 \arrow[d, "x_0"] \\ Y \arrow[r, "f"] & X \end{tikzcd} \tag{4.13}$$

be a Cartesian diagram in $\tilde{\mathcal{C}}''$, where x_0 and hence y_0 are atlases. Let $\mathrm{N}(\mathbf{\Delta}_+^{op}) \times \Delta^1 \to \tilde{\mathcal{C}}''$ be a Čech nerve, as shown in the following diagram

$$\begin{tikzcd} Y_\bullet \arrow[r, "f_\bullet"] \arrow[d, "y_\bullet"'] & X_\bullet \arrow[d, "x_\bullet"] \\ Y \arrow[r, "f"] & X. \end{tikzcd} \tag{4.14}$$

We call such a diagram a *simplicial Cartesian atlas* of f. We have $\dim^+(f_n) = \dim^+(f)$. By ${}_{\tilde{\mathcal{C}}'}\mathrm{EO}_!^*$, we have

$$x_0^* f_! \lambda_Y \langle d \rangle \simeq f_{0!} y_0^* \lambda_Y \langle d \rangle \simeq f_{0!} \lambda_Y \langle d \rangle \in \mathcal{D}^{\leq 0}(X, \lambda),$$

which implies that $f_! \lambda_Y \langle d \rangle$ is in $\mathcal{D}^{\leq 0}(X, \lambda)$ by the definition of the t-structure. The uniqueness of the trace map follows from condition (2) of Remark 3.3.2 applied to the diagram (4.13) and (P3) applied to x_0.

For $n \geq 0$, we have trace maps $\mathrm{Tr}_{f_n} \colon \tau^{\geq 0} f_{n!} \lambda_{Y_n} \langle d \rangle \to \lambda_{X_n}$. By condition (2) applied to the squares induced by $f_\bullet$, $\tau^{\leq 0} x_{\bullet *} \mathrm{Tr}_{f_\bullet}$ is a morphism of cosimplicial objects of $\mathcal{D}^{\heartsuit}(X, \lambda)$. Taking limit, we obtain

$$\varprojlim_{n \in \mathbf{\Delta}} \tau^{\leq 0} x_{n*} \mathrm{Tr}_{f_n} \colon \varprojlim_{n \in \mathbf{\Delta}} \tau^{\leq 0} x_{n*} \tau^{\geq 0} f_{n!} \lambda_{Y_n} \langle d \rangle \to \varprojlim_{n \in \mathbf{\Delta}} \tau^{\leq 0} x_{n*} \lambda_{X_n} \simeq \lambda_X.$$

However, the left-hand side is isomorphic to

$$\varprojlim_{n \in \mathbf{\Delta}} \tau^{\leq 0} x_{n*} \tau^{\geq 0} f_{n!} y_n^* \lambda_Y \langle d \rangle \simeq \varprojlim_{n \in \mathbf{\Delta}} \tau^{\leq 0} x_{n*} \tau^{\geq 0} x_n^* f_! \lambda_Y \langle d \rangle \simeq \varprojlim_{n \in \mathbf{\Delta}} \tau^{\leq 0} x_{n*} x_n^* \tau^{\geq 0} f_! \lambda_Y \langle d \rangle \simeq \tau^{\geq 0} f_! \lambda_Y \langle d \rangle.$$

Therefore, we obtain a map $\mathrm{Tr}_{f_\bullet} \colon \tau^{\geq 0} f_! \lambda_Y \langle d \rangle \to \lambda_X$.

This extends the trace map of the input. In fact, for f in $\mathcal{C}_1''$, by condition (2) applied to (4.14), $\mathrm{Tr}_{f_\bullet}$ can be identified with $\varprojlim_{n \in \mathbf{\Delta}} x_{n*} x_n^* \mathrm{Tr}_f$. Moreover, condition (2) holds in general if one interprets Tr_f as $\mathrm{Tr}_{f_\bullet}$ and $\mathrm{Tr}_{f'}$ as $\mathrm{Tr}_{f'_\bullet}$, where $f'_\bullet$ is a simplicial Cartesian atlas of f', compatible with $f_\bullet$. In fact, by condition (2) for the input, the bottom square of the diagram

$$\begin{tikzcd} u^* \tau^{\geq 0} f_! \lambda_Y \langle d \rangle \arrow[rr, "u^* \mathrm{Tr}_{f_\bullet}"] \arrow[dr, "\sim"] \arrow[dd, "\simeq"'] & & u^* \lambda_X \arrow[dr, "\sim"] \arrow[dd, "\simeq"] & \\ & \tau^{\geq 0} f'_! \lambda_{Y'} \langle d \rangle \arrow[rr, "\mathrm{Tr}_{f'_\bullet}"] \arrow[dd, "\simeq"] & & \lambda_{X'} \arrow[dd, "\simeq"] \\ \varprojlim \tau^{\leq 0} x'_{n*} u_n^* \tau^{\geq 0} f_{n!} \lambda_{Y_n} \langle d \rangle \arrow[rr, "\varprojlim \tau^{\leq 0} x'_{n*} u_n^* \mathrm{Tr}_{f_n}"] \arrow[dr, "\sim"] & & \varprojlim \tau^{\leq 0} x'_{n*} u_n^* \lambda_{X_n} \arrow[dr, "\sim"] & \\ & \varprojlim \tau^{\leq 0} x'_{n*} \tau^{\geq 0} f'_{n!} \lambda_{Y'_n} \arrow[rr, "\varprojlim \tau^{\leq 0} x'_{n*} \mathrm{Tr}_{f'_n}"] & & \varprojlim \tau^{\leq 0} x'_{n*} \lambda_{X'_n} \end{tikzcd}$$

is commutative, where all the limits are taken over $n \in \mathbf{\Delta}$. Since the vertical squares are commutative, it follows that the top square is commutative as well. The case of condition (2) where u is an atlas then implies that $\mathrm{Tr}_{f_\bullet}$ does not depend on the choice of $f_\bullet$. We may therefore denote it by Tr_f.

It remains to check conditions (1) and (3) of Remark 3.3.2. Similarly to the situation of condition (2), these follow from the input by taking limits. □

Lemma 4.3.10. *If $f\colon Y\to X$ is in $\mathcal{R}\cap\tilde{\mathcal{E}}''_d\cap\tilde{\mathcal{C}}''_1$, the induced natural transformation*

$$f^*\langle d\rangle=\mathrm{id}_Y\circ f^*\langle d\rangle\to f^!\circ f_!\circ f^*\langle d\rangle\xrightarrow{f^!\circ u_f}f^!$$

is an equivalence, where the first arrow is given by the unit transformation.

Proof. Consider diagram (4.14). We need to show that for every object $\mathscr{K}$ of $\mathcal{D}(X,\lambda)$, the natural map $f^*\mathscr{K}\langle d\rangle\to f^!\mathscr{K}$ is an equivalence. By Proposition 4.3.4 (1), the map $\mathscr{K}\to\varprojlim_{n\in\mathbf{\Delta}}u_{n*}u_n^*\mathscr{K}$ is an equivalence. Moreover, $f^!$ preserves small limits, and, by (P5$^{\mathrm{bis}}$) (1), so does f^*, since f is smooth. Therefore, we may assume $\mathscr{K}=x_{n*}\mathscr{L}$, where $\mathscr{L}\in\mathcal{D}(X_n,\lambda)$. Similarly to (4.5), the diagram

$$\begin{tikzcd}
f^*x_{n*}\mathscr{L}\langle d\rangle \arrow[r] \arrow[d] & y_{n*}f_n^*\mathscr{L}\langle d\rangle \arrow[d] \\
f^!x_{n*}\mathscr{L} \arrow[r] & y_{n*}f_n^!\mathscr{L}
\end{tikzcd}$$

is commutative up to homotopy. The upper horizontal arrow is an equivalence by (P5$^{\mathrm{bis}}$) (1), the lower horizontal arrow is an equivalence by ${}_{\tilde{\mathcal{C}}'}\mathrm{EO}^*_!$, and the right vertical arrow is an equivalence by (P6) for the input. It follows that the left vertical arrow is an equivalence. ☐

Lemma 4.3.11 (P7 (1))**.** *There exists a unique way to define the trace map*

$$\mathrm{Tr}_f\colon\tau^{\geq 0}f_!\lambda_Y\langle d\rangle\to\lambda_X,$$

for morphisms $f\colon Y\to X$ in $\tilde{\mathcal{E}}_{\mathrm{t}}\cap\tilde{\mathcal{C}}''_1$ and integers $d\geq\dim^+(f)$, satisfying (P7) (1) and extending the input. In particular, for such a morphism f, $f_!\lambda_Y\langle d\rangle$ is in $\mathcal{D}^{\leq 0}(X,\lambda)$.

Proof. Let $Y_\bullet\colon\mathrm{N}(\mathbf{\Delta}_+^{op})\to\tilde{\mathcal{C}}'$ be a Čech nerve of an atlas $y_0\colon Y_0\to Y$, and form a triangle

(4.15)
$$\begin{tikzcd}
 & Y \arrow[dr, "f"] & \\
Y_\bullet \arrow[ur, "y_\bullet"] \arrow[rr, "f_\bullet"] & & X.
\end{tikzcd}$$

For $n\geq 0$, f_n is in $\mathcal{R}\cap\tilde{\mathcal{E}}_{\mathrm{t}}\cap\tilde{\mathcal{C}}''_1$. By Proposition 4.3.4 (2), we have equivalences

$$\varinjlim_{n\in\mathbf{\Delta}^{op}}f_{n!}y_n^!\lambda_Y\simeq\varinjlim_{n\in\mathbf{\Delta}^{op}}f_!y_{n!}y_n^!\lambda_Y\xrightarrow{\sim}f_!\varinjlim_{n\in\mathbf{\Delta}^{op}}y_{n!}y_n^!\lambda_Y\xrightarrow{\sim}f_!\lambda_Y.$$

Since y_n is in $\mathcal{R}\cap\tilde{\mathcal{E}}''\cap\tilde{\mathcal{C}}''_1$, by Lemmas 4.3.10 and Remark 4.1.5 (4), we have equivalences

$$\varinjlim_{n\in\mathbf{\Delta}^{op}}f_{n!}\lambda_{Y_n}\langle d+\dim y_n\rangle\simeq\varinjlim_{n\in\mathbf{\Delta}^{op}}f_{n!}y_n^*\lambda_Y\langle d+\dim y_n\rangle\xrightarrow{\sim}\varinjlim_{n\in\mathbf{\Delta}^{op}}f_{n!}y_n^!\lambda_Y\langle d\rangle.$$

Combining the above, we obtain an equivalence $\varinjlim_{n\in\mathbf{\Delta}^{op}}f_{n!}\lambda_{Y_n}\langle d+\dim y_n\rangle\xrightarrow{\sim}f_!\lambda_Y\langle d\rangle$. By Lemma 4.3.9, each $f_{n!}\lambda_{Y_n}\langle d+\dim y_n\rangle$ is in $\mathcal{D}^{\leq 0}(X,\lambda)$. It follows that the colimit is as well by [30, 1.2.1.6]. Moreover, the composite map

$$\tau^{\geq 0}f_{n!}\lambda_{Y_n}\langle d+\dim y_n\rangle\to\varinjlim_{n\in\mathbf{\Delta}^{op}}\tau^{\geq 0}f_{n!}\lambda_{Y_n}\langle d+\dim y_n\rangle\xrightarrow{\sim}\tau^{\geq 0}\varinjlim_{n\in\mathbf{\Delta}^{op}}f_{n!}\lambda_{Y_n}\langle d+\dim y_n\rangle\xrightarrow{\sim}\tau^{\geq 0}f_!\lambda_Y\langle d\rangle$$

is induced by Tr_{f_n}. The uniqueness of Tr_f then follows from condition (3) in Remark 3.3.2 applied to the triangle (4.15).

Condition (3) applied to the triangles induced by $f_\bullet$ implies the compatibility of

$$\mathrm{Tr}_{f_n}\colon\tau^{\geq 0}f_{n!}\lambda_{Y_n}\langle d+\dim y_n\rangle\to\lambda_X$$

with the transition maps, so that we obtain a map $\mathrm{Tr}_{f_\bullet}\colon\tau^{\geq 0}f_!\lambda_Y\langle d\rangle\to\lambda_X$. This extends the trace map of Lemma 4.3.9, by condition (3) applied to (4.15) for f representable. Moreover, condition (3) holds

for g representable, if we interpret Tr_f as $\mathrm{Tr}_{f_\bullet}$ and Tr_h as $\mathrm{Tr}_{h_\bullet}$, where $h_\bullet\colon Y_\bullet \times_Y Z \to X$. In fact, by condition (3) for representable morphisms, the diagram

$$\begin{tikzcd}
\varinjlim \tau^{\geq 0} f_{n!}(\tau^{\geq 0} g_! \lambda_Z\langle e\rangle)\langle d+\dim y_n\rangle \arrow[rr, "\varinjlim \tau^{\geq 0} f_{n!}\mathrm{Tr}_g\langle d+\dim y_n\rangle"] \arrow[d] \arrow[dr, "\sim"] & & \varinjlim \tau^{\geq 0} f_! \lambda_Y\langle d\rangle \arrow[dr, "\sim"] & \\
\varinjlim \tau^{\geq 0} h_{n!}\lambda_Z\langle d+e+\dim y_n\rangle \arrow[dr, "\sim"] & \tau^{\geq 0} f_!(\tau^{\geq 0} g_!\lambda_Z\langle e\rangle)\langle d\rangle \arrow[rr, "\tau^{\geq 0} f_! \mathrm{Tr}_g\langle d\rangle"] \arrow[d, "\simeq"'] & & \tau^{\geq 0} f_! \lambda_Y\langle d\rangle \arrow[d, "\mathrm{Tr}_{f_\bullet}"] \\
& \tau^{\geq 0} h_! \lambda_Z\langle d+e\rangle \arrow[rr, "\mathrm{Tr}_{h_\bullet}"] & & \lambda_X
\end{tikzcd}$$

commutes, where all the colimits are taken over $n \in \mathbf{\Delta}^{op}$. It follows that $\mathrm{Tr}_{f_\bullet}$ does not depend on the choice of $f_\bullet$. We may therefore denote it by Tr_f.

It remains to check the functoriality of the trace map. Similarly to the above special case of condition (2), this follows from the functoriality of the trace map for representable morphisms by taking colimits. $\square$

Proposition 4.3.12 (P7 (2))**.** *If $f\colon Y \to X$ is in $\tilde{\mathcal{E}}''_d \cap \tilde{\mathcal{C}}''_1$, the induced natural transformation*

$$f^*\langle d\rangle = \mathrm{id}_Y \circ f^*\langle d\rangle \to f^! \circ f_! \circ f^*\langle d\rangle \xrightarrow{f^! \circ u_f} f^!$$

is an equivalence, where the first arrow is given by the unit transformation.

Proof. We need to show that $f^*\mathscr{K}\langle d\rangle \to f^!\mathscr{K}$ is an equivalence of every object $\mathscr{K}$ of $\mathcal{D}(X,\lambda)$. Let $y_0\colon Y_0 \to Y$ be an atlas. Since v_0^* is conservative by Lemma 4.3.3, we only need to show that the composite map

$$y_0^*\mathscr{K}\langle \dim f_0\rangle \xrightarrow{\sim} y_0^* f^*\mathscr{K}\langle d+\dim y_0\rangle \to y_0^* f^!\mathscr{K}\langle \dim y_0\rangle \xrightarrow{\sim} y_0^! f^!\mathscr{K} \xrightarrow{\sim} f_0^!\mathscr{K}$$

is an equivalence, where $f_0\colon Y_0 \to X$ is a composite of f and y_0. However, this follows from Lemma 4.3.10 applied to f_0. $\square$

5. Running DESCENT

In this chapter, we run the program DESCENT recursively to construct the theory of six operations of quasi-separated schemes in §5.1, algebraic spaces in §5.2, (classical) Artin stacks in §5.3, and eventually higher Artin stacks in §5.4. Moreover, we start from algebraic spaces to construct the theory for higher Deligne–Mumford stacks as well in §5.5. We would like to point out that although higher DM stacks are special cases of higher Artin stacks, we have less restrictions on the coefficient rings for the former.

5.1. **Quasi-separated schemes.** Recall that $\mathcal{S}\mathrm{ch}^{\mathrm{qs}}$ is the full subcategory of $\mathcal{S}\mathrm{ch}$ spanned by quasi-separated schemes, which contains $\mathcal{S}\mathrm{ch}^{\mathrm{qc.sep}}$ as a full subcategory. We run the program DESCENT with the input data in Example 4.1.10. Then the output consists of the following two maps:

$$(5.1)\qquad {}_{\mathcal{S}\mathrm{ch}^{\mathrm{qs}}}\mathrm{EO}\colon \delta^*_{2,\{2\}}\mathrm{Fun}(\Delta^1, \mathrm{N}(\mathcal{S}\mathrm{ch}^{\mathrm{qs}}))^{\mathrm{cart}}_{F^0,A^\to} \to \mathrm{Fun}(\mathrm{N}(\mathcal{R}\mathrm{ind}^{op}_{\mathrm{tor}}), \mathrm{Mon}^{\mathcal{P}\mathrm{r}^{\mathrm{L}}_{\mathrm{st}}}_{\mathcal{P}\mathrm{f}}(\mathcal{C}\mathrm{at}_\infty)),$$

$$(5.2)\qquad {}_{\mathcal{S}\mathrm{ch}^{\mathrm{qs}}}\mathrm{EO}^*_\otimes\colon \mathrm{N}(\mathcal{S}\mathrm{ch}^{\mathrm{qs}})^{op} \to \mathrm{Fun}(\mathrm{N}(\mathcal{R}\mathrm{ind}^{op}), \mathcal{P}\mathrm{r}^{\mathrm{L}\otimes}_{\mathrm{st,cl}})$$

and Output II. Here F (resp. A) denotes the set of morphisms locally of finite type (resp. all morphisms) of quasi-separated schemes.

For each object X of $\mathcal{S}\mathrm{ch}^{\mathrm{qs}}$, we denote by $\mathrm{\acute{E}t}^{\mathrm{qs}}(X)$ the quasi-separated étale site of X. Its underlying category is the full subcategory of $\mathcal{S}\mathrm{ch}^{\mathrm{qs}}_{/X}$ spanned by étale morphisms. We denote by $X_{\mathrm{qs.\acute{e}t}}$ the associated topos, namely the category of sheaves on $\mathrm{\acute{E}t}^{\mathrm{qs}}(X)$. For every object X of $\mathcal{S}\mathrm{ch}^{\mathrm{qc.sep}}$, the inclusions $\mathrm{\acute{E}t}^{\mathrm{qc.sep}}(X) \subseteq \mathrm{\acute{E}t}^{\mathrm{qs}}(X) \subseteq \mathrm{\acute{E}t}(X)$ induce an equivalences of topoi $X_{\mathrm{qc.sep.\acute{e}t}} \to X_{\mathrm{qs.\acute{e}t}} \to X_{\mathrm{\acute{e}t}}$.

As in (3.7), the pseudofunctor $\mathcal{S}\mathrm{ch}^{\mathrm{qs}} \times \mathcal{R}\mathrm{ind} \to \mathcal{R}\mathrm{inged}\mathcal{P}\mathcal{T}\mathrm{opos}$ sending $(X,(\Xi,\Lambda))$ to $(X^{\Xi}_{\mathrm{qs.\acute{e}t}},\Lambda)$ induces a map $\mathrm{N}(\mathcal{S}\mathrm{ch}^{\mathrm{qs}}) \times \mathrm{N}(\mathcal{R}\mathrm{ind}) \to \mathrm{N}(\mathcal{R}\mathrm{inged}\mathcal{P}\mathcal{T}\mathrm{opos})$. Composing with $\mathbf{T}^\otimes$ (2.1), we obtain

$$(5.3)\qquad {}^{\mathrm{qs.\acute{e}t}}_{\mathcal{S}\mathrm{ch}^{\mathrm{qs}}}\mathrm{EO}^*_\otimes\colon \mathrm{N}(\mathcal{S}\mathrm{ch}^{\mathrm{qs}})^{op} \to \mathrm{Fun}(\mathrm{N}(\mathcal{R}\mathrm{ind}^{op}), \mathcal{P}\mathrm{r}^{\mathrm{L}\otimes}_{\mathrm{st,cl}})$$

such that the restriction ${}^{\mathrm{qs.\acute{e}t}}_{\mathcal{S}\mathrm{ch}^{\mathrm{qs}}}\mathrm{EO}^*_\otimes \mid \mathrm{N}(\mathcal{S}\mathrm{ch}^{\mathrm{qc.sep}})^{op}$ is equivalent to ${}_{\mathcal{S}\mathrm{ch}^{\mathrm{qc.sep}}}\mathrm{EO}^*_\otimes$. By the same proof of Proposition 3.3.5 (1), we have the following.

Proposition 5.1.1 (Cohomological descent for étale topoi)**.** *Let $f\colon Y \to X$ be smooth surjective morphism of quasi-separated schemes. Then f is of universal ${}^{\mathrm{qs.\acute{e}t}}_{\mathcal{S}\mathrm{ch}^{\mathrm{qs}}}\mathrm{EO}^*_\otimes$-descent.*

From the above proposition and Proposition 4.1.1, we obtain the following compatibility result.

Proposition 5.1.2. *The two maps ${}_{\mathcal{S}\mathrm{ch}^{\mathrm{qs}}}\mathrm{EO}^*_\otimes$ (5.2) and ${}^{\mathrm{qs.\acute{e}t}}_{\mathcal{S}\mathrm{ch}^{\mathrm{qs}}}\mathrm{EO}^*_\otimes$ (5.3) are equivalent.*

Remark 5.1.3. Let $\lambda = (\Xi, \Lambda)$ be an object of $\mathcal{R}\mathrm{ind}$. Then it is easy to see that the usual t-structure on $\mathcal{D}(\mathcal{X}^{\Xi}_{\mathrm{qs.\acute{e}t}}, \Lambda)$ coincides with the one on $\mathcal{D}(\mathcal{X}, \lambda)$ obtained in the output of the program DESCENT.

5.2. **Algebraic spaces.** Let $\mathcal{E}\mathrm{sp}$ be the category of algebraic spaces (§0.1). It contains $\mathcal{S}\mathrm{ch}^{\mathrm{qs}}$ as a full subcategory. We run the program DESCENT with the following input:

- $\tilde{\mathcal{C}} = \mathrm{N}(\mathcal{E}\mathrm{sp})$. It is geometric.
- $\mathcal{C} = \mathrm{N}(\mathcal{S}\mathrm{ch}^{\mathrm{qs}})$, and $\mathsf{s}'' \to \mathsf{s}'$ is the unique morphism $\operatorname{Spec} \mathbb{Z}[\mathsf{L}^{-1}] \to \operatorname{Spec} \mathbb{Z}$. In particular, $\mathcal{C}' = \mathcal{C}$ and $\tilde{\mathcal{C}}' = \tilde{\mathcal{C}}$.
- $\tilde{\mathcal{E}}_{\mathrm{s}}$ is the set of *surjective* morphisms of algebraic spaces.
- $\tilde{\mathcal{E}}'$ is the set of *étale* morphisms of algebraic spaces.
- $\tilde{\mathcal{E}}''$ is the set of *smooth* morphisms of algebraic spaces.
- $\tilde{\mathcal{E}}''_d$ is the set of *smooth* morphisms of algebraic spaces of pure relative dimension d. In particular, $\tilde{\mathcal{E}}' = \tilde{\mathcal{E}}''_0$.
- $\tilde{\mathcal{E}}_{\mathrm{t}}$ is the set of *flat* morphisms *locally of finite presentation* of algebraic spaces.
- $\tilde{\mathcal{F}} = F$ is the set of morphisms locally of finite type of algebraic spaces.
- $\mathcal{L} = \mathrm{N}(\mathcal{R}\mathrm{ind}^{op})$, $\mathcal{L}' = \mathrm{N}(\mathcal{R}\mathrm{ind}^{op}_{\mathrm{tor}})$, and $\mathcal{L}'' = \mathrm{N}(\mathcal{R}\mathrm{ind}^{op}_{\mathsf{L}\text{-tor}})$.
- $\dim^+$ is the upper relative dimension (Definition 4.1.9).
- Input I and II is the output of §5.1. In particular, ${}_{\mathcal{C}'}\mathrm{EO}$ is (5.1), and ${}_{\mathcal{C}}\mathrm{EO}^*_\otimes$ is (5.2).

Then the output consists of the following two maps:

$$ {}_{\mathcal{E}\mathrm{sp}}\mathrm{EO}\colon \delta^*_{2,\{2\}}\mathrm{Fun}(\Delta^1, \mathrm{N}(\mathcal{E}\mathrm{sp}))^{\mathrm{cart}}_{F^0, A^{\to}} \to \mathrm{Fun}(\mathrm{N}(\mathcal{R}\mathrm{ind}^{op}_{\mathrm{tor}}), \mathrm{Mon}^{\mathcal{P}\mathrm{r}^{\mathrm{L}}_{\mathrm{st}}}_{\mathcal{P}\mathrm{f}}(\mathcal{C}\mathrm{at}_\infty)), \tag{5.4} $$

$$ {}_{\mathcal{E}\mathrm{sp}}\mathrm{EO}^*_\otimes\colon \mathrm{N}(\mathcal{E}\mathrm{sp})^{op} \to \mathrm{Fun}(\mathrm{N}(\mathcal{R}\mathrm{ind}^{op}), \mathcal{P}\mathrm{r}^{\mathrm{L}\otimes}_{\mathrm{st,cl}}) \tag{5.5} $$

and Output II. Here F (resp. A) denotes the set of morphisms locally of finite type (resp. all morphisms) of algebraic spaces.

For each object X of $\mathcal{E}\mathrm{sp}$, we denote by $\mathrm{\acute{E}t}^{\mathrm{esp}}(X)$ the spatial étale site of X. Its underlying category is the full subcategory of $\mathcal{E}\mathrm{sp}_{/X}$ spanned by étale morphisms. We denote by $X_{\mathrm{esp.\acute{e}t}}$ the associated topos, namely the category of sheaves on $\mathrm{\acute{E}t}^{\mathrm{esp}}(X)$. For every object X of $\mathcal{S}\mathrm{ch}^{\mathrm{qs}}$, the inclusion of the original étale site $\mathrm{\acute{E}t}^{\mathrm{qs}}(X)$ of X into $\mathrm{\acute{E}t}^{\mathrm{esp}}(X)$ induces an equivalence of topoi $X_{\mathrm{esp.\acute{e}t}} \to X_{\mathrm{qs.\acute{e}t}}$.

As in 5.1, we have a map

$$ {}^{\mathrm{esp.\acute{e}t}}_{\mathcal{E}\mathrm{sp}}\mathrm{EO}^*_\otimes\colon \mathrm{N}(\mathcal{E}\mathrm{sp})^{op} \to \mathrm{Fun}(\mathrm{N}(\mathcal{R}\mathrm{ind}^{op}), \mathcal{P}\mathrm{r}^{\mathrm{L}\otimes}_{\mathrm{st,cl}}) \tag{5.6} $$

such that the restriction ${}^{\mathrm{esp.\acute{e}t}}_{\mathcal{E}\mathrm{sp}}\mathrm{EO}^*_\otimes | \mathrm{N}(\mathcal{S}\mathrm{ch}^{\mathrm{qs}})^{op}$ is equivalent to ${}_{\mathcal{S}\mathrm{ch}^{\mathrm{qs}}}\mathrm{EO}^*_\otimes$. Moreover, we have the following results.

Proposition 5.2.1 (Cohomological descent for étale topoi)**.** *Let $f\colon Y \to X$ be a smooth surjective morphism of algebraic spaces. Then f is of universal ${}^{\mathrm{esp.\acute{e}t}}_{\mathcal{E}\mathrm{sp}}\mathrm{EO}^*_\otimes$-descent.*

Proposition 5.2.2. *The two maps ${}_{\mathcal{E}\mathrm{sp}}\mathrm{EO}^*_\otimes$ (5.5) and ${}^{\mathrm{esp.\acute{e}t}}_{\mathcal{E}\mathrm{sp}}\mathrm{EO}^*_\otimes$ (5.6) are equivalent.*

Remark 5.2.3. Let $\lambda = (\Xi, \Lambda)$ be an object of $\mathcal{R}\mathrm{ind}$. Then the usual t-structure on $\mathcal{D}(\mathcal{X}^{\Xi}_{\mathrm{esp.\acute{e}t}}, \Lambda)$ coincides with the one on $\mathcal{D}(\mathcal{X}, \lambda)$ obtained in the output of the program.

In our construction of the map (3.5) in §3.2, the essential facts we used from algebraic geometry are Nagata's compactification and proper base change. Nagata's compactification has been extended to separated morphisms of finite type between quasi-compact and quasi-separated algebraic spaces [8, 1.2.1].

Proper base change for algebraic spaces follows from the case of schemes by cohomological descent and Chow's lemma for algebraic spaces [34, I 5.7.13] or the existence theorem of a finite cover by a scheme. The latter is a special case of [36, Theorem B] and also follows from the Noetherian case [26, 16.6] by Noetherian approximation of algebraic spaces [8, 1.2.2].

Therefore, if we denote by $\mathcal{E}\mathrm{sp}^{\mathrm{qc.sep}}$ the full subcategory of $\mathcal{E}\mathrm{sp}$ spanned by (small) disjoint union of quasi-compact and separated algebraic spaces (hence contains $\mathcal{S}\mathrm{ch}^{\mathrm{qc.sep}}$ as a full subcategory), and repeat the process in §3.2, we obtain a map

$$_{\mathcal{E}\mathrm{sp}^{\mathrm{qc.sep}}}^{\mathrm{var}}\mathrm{EO}\colon \delta^*_{2,\{2\}}\mathrm{Fun}(\Delta^1,\mathrm{N}(\mathcal{E}\mathrm{sp}^{\mathrm{qc.sep}}))^{\mathrm{cart}}_{F^0,A^{\rightarrow}} \to \mathrm{Fun}(\mathrm{N}(\mathcal{R}\mathrm{ind}^{op}_{\mathrm{tor}}),\mathrm{Mon}^{\mathcal{P}\mathrm{r}^{\mathrm{L}}_{\mathrm{st}}}_{\mathcal{P}\mathrm{f}}(\mathcal{C}\mathrm{at}_\infty)).$$

Its restriction to $\delta^*_{2,\{2\}}\mathrm{Fun}(\Delta^1,\mathrm{N}(\mathcal{S}\mathrm{ch}^{\mathrm{qc.sep}}))^{\mathrm{cart}}_{F^0,A^{\rightarrow}}$ is equivalent to $_{\mathcal{S}\mathrm{ch}^{\mathrm{qc.sep}}}\mathrm{EO}$.

Proposition 5.2.4. *The restriction* $_{\mathcal{E}\mathrm{sp}}\mathrm{EO}\mid\delta^*_{2,\{2\}}\mathrm{Fun}(\Delta^1,\mathrm{N}(\mathcal{E}\mathrm{sp}^{\mathrm{qc.sep}}))^{\mathrm{cart}}_{F^0,A^{\rightarrow}}$ *is equivalent to the map* $_{\mathcal{E}\mathrm{sp}^{\mathrm{qc.sep}}}^{\mathrm{var}}\mathrm{EO}$.

Proof. By Remark 4.1.8 (2), it suffices to prove that $_{\mathcal{E}\mathrm{sp}^{\mathrm{qc.sep}}}^{\mathrm{var}}\mathrm{EO}$ satisfies (P4). For this, we can repeat the proof of 3.3.5. The analogue of Remark 3.3.4 holds for algebraic spaces because the definition of trace maps is local for the étale topology on target. □

5.3. **Artin Stacks.** Let $\mathcal{C}\mathrm{hp}$ be the $(2,1)$-category of Artin stacks (§0.1). It contains $\mathcal{E}\mathrm{sp}$ as a full subcategory. We run the *simplified* DESCENT (see Variant 4.1.7) with the following input:

- $\tilde{\mathcal{C}}=\mathrm{N}(\mathcal{C}\mathrm{hp})$. It is geometric.
- $\mathcal{C}=\mathrm{N}(\mathcal{E}\mathrm{sp})$, and $\mathsf{s}''\to\mathsf{s}'$ is the identity morphism of $\mathrm{Spec}\,\mathbb{Z}[\mathsf{L}^{-1}]$. In particular, $\mathcal{C}'=\mathcal{C}''=\mathrm{N}(\mathcal{E}\mathrm{sp}_{\mathsf{L}})$ (resp. $\tilde{\mathcal{C}}'=\tilde{\mathcal{C}}''=\mathrm{N}(\mathcal{C}\mathrm{hp}_{\mathsf{L}})$), where $\mathcal{E}\mathrm{sp}_{\mathsf{L}}$ (resp. $\mathcal{C}\mathrm{hp}_{\mathsf{L}}$) is the category of L-coprime algebraic spaces (resp. Artin stacks).
- $\tilde{\mathcal{E}}_{\mathrm{s}}$ is the set of *surjective* morphisms of Artin stacks.
- $\tilde{\mathcal{E}}'=\tilde{\mathcal{E}}''$ is the set of *smooth* morphisms of Artin stacks.
- $\tilde{\mathcal{E}}''_d$ is the set of *smooth* morphisms of Artin stacks of pure relative dimension d.
- $\tilde{\mathcal{E}}_{\mathrm{t}}$ is the set of *flat* morphisms *locally of finite presentation* of Artin stacks.
- $\tilde{\mathcal{F}}=F$ is the set of morphisms *locally of finite type* of Artin stacks.
- $\mathcal{L}=\mathrm{N}(\mathcal{R}\mathrm{ind}^{op})$, and $\mathcal{L}'=\mathcal{L}''=\mathrm{N}(\mathcal{R}\mathrm{ind}^{op}_{\mathsf{L}\text{-tor}})$.
- $\dim^+$ is upper relative dimension, which is defined as a special case in Definition 5.4.4.
- Input I and II is given by the output of §5.2. In particular, $_{\mathcal{C}}\mathrm{EO}^*_{\otimes}$ is (5.5), and

$$_{\mathcal{C}}\mathrm{EO}={}_{\mathcal{E}\mathrm{sp}_{\mathsf{L}}}\mathrm{EO}\colon \delta^*_{2,\{2\}}\mathrm{Fun}(\Delta^1,\mathrm{N}(\mathcal{E}\mathrm{sp}_{\mathsf{L}}))^{\mathrm{cart}}_{F^0,A^{\rightarrow}} \to \mathrm{Fun}(\mathrm{N}(\mathcal{R}\mathrm{ind}^{op}_{\mathsf{L}\text{-tor}}),\mathrm{Mon}^{\mathcal{P}\mathrm{r}^{\mathrm{L}}_{\mathrm{st}}}_{\mathcal{P}\mathrm{f}}(\mathcal{C}\mathrm{at}_\infty))$$

  is the map induced from (5.4) by restricting to $\mathcal{E}\mathrm{sp}_{\mathsf{L}}$ and $\mathcal{R}\mathrm{ind}_{\mathsf{L}\text{-tor}}$.

Then the output consists of the following two maps:

$$_{\mathcal{C}\mathrm{hp}_{\mathsf{L}}}\mathrm{EO}\colon \delta^*_{2,\{2\}}\mathrm{Fun}(\Delta^1,\mathrm{N}(\mathcal{C}\mathrm{hp}_{\mathsf{L}}))^{\mathrm{cart}}_{F^0,A^{\rightarrow}} \to \mathrm{Fun}(\mathrm{N}(\mathcal{R}\mathrm{ind}^{op}_{\mathsf{L}\text{-tor}}),\mathrm{Mon}^{\mathcal{P}\mathrm{r}^{\mathrm{L}}_{\mathrm{st}}}_{\mathcal{P}\mathrm{f}}(\mathcal{C}\mathrm{at}_\infty)), \tag{5.7}$$

$$_{\mathcal{C}\mathrm{hp}}\mathrm{EO}^*_{\otimes}\colon \mathrm{N}(\mathcal{C}\mathrm{hp})^{op}\to\mathrm{Fun}(\mathrm{N}(\mathcal{R}\mathrm{ind}^{op}),\mathcal{P}\mathrm{r}^{\mathrm{L}\otimes}_{\mathrm{st,cl}}) \tag{5.8}$$

and Output II. Here F (resp. A) denotes the set of morphisms locally of finite type (all morphisms) of Artin stacks.

Let us recall the lisse-étale site $\text{Lis-ét}(X)$ of an Artin stack X. Its underlying category, the full subcategory (which is in fact an ordinary category) of $\mathcal{C}\mathrm{hp}_{/X}$ spanned by smooth morphisms whose sources are algebraic spaces, is equivalent to a $\mathcal{U}$-small category. In particular, $\text{Lis-ét}(X)$ endowed with the étale topology is a $\mathcal{U}$-site. We denote by $X_{\text{lis-ét}}$ the associated topos. Let $M\subseteq\mathrm{N}(\mathcal{C}\mathrm{hp})_1$ be the set of smooth representable morphisms between Artin stacks. The lisse-étale topos has enough points by [26, 12.2.2], and is functorial with respect to M, so that we obtain a functor $\mathcal{C}\mathrm{hp}\times\mathcal{R}\mathrm{ind}\to\mathcal{R}\mathrm{inged}\mathcal{P}\mathcal{T}\mathrm{opos}$. Composing with $\mathbf{T}^{\otimes}$, we obtain a functor $\mathrm{N}(\mathcal{C}\mathrm{hp})^{op}_M\times\mathrm{N}(\mathcal{R}\mathrm{ind})^{op}\to\mathcal{P}\mathrm{r}^{\mathrm{L}\otimes}_{\mathrm{st,cl}}$ sending $(X,(\Xi,\Lambda))$ to $\mathcal{D}(X^{\Xi}_{\text{lis-ét}},\Lambda)^{\otimes}$.

To simplify notations, for an algebraic space U, we will write $U_{\text{ét}}$ instead of $U_{\text{esp.ét}}$ in what follows. We let $\mathcal{D}_{\mathrm{cart}}(X^{\Xi}_{\text{lis-ét}},\Lambda)\subseteq\mathcal{D}_{(}X^{\Xi}_{\text{lis-ét}},\Lambda)$ be the full subcategory consisting of complexes whose cohomology sheaves are all Cartesian (§0.1), or, equivalently, complexes K such that for every morphism $f\colon Y'\to Y$ of $\text{Lis-ét}(X)$, the map $f^*(K\mid Y_{\text{ét}})\to(K\mid Y'_{\text{ét}})$ is an equivalence. The full subcategory is stable under

tensor product and contains the monoidal unit, so that it defines a symmetric monoidal ∞-category $\mathcal{D}_{\mathrm{cart}}(X^{\Xi}_{\text{lis-ét}},\Lambda)^{\otimes}$. Replacing $\mathcal{D}(X^{\Xi}_{\text{lis-ét}},\Lambda)^{\otimes}$ by $\mathcal{D}_{\mathrm{cart}}(X^{\Xi}_{\text{lis-ét}},\Lambda)^{\otimes}$ in the above functor, we obtain a functor

$$^{\text{lis-ét}}_{\mathcal{C}\mathrm{hp}}\mathrm{EO}^*_{\otimes}\colon \mathrm{N}(\mathcal{C}\mathrm{hp})^{op}_{M}\times \mathrm{N}(\mathcal{R}\mathrm{ind})^{op}\to \mathcal{C}\mathrm{at}^{\otimes}_{\infty}.$$

Let $M'=M\cap \mathrm{Ar}(\mathcal{E}\mathrm{sp})$. The restriction $^{\text{lis-ét}}_{\mathcal{C}\mathrm{hp}}\mathrm{EO}^*_{\otimes}\mid \mathrm{N}(\mathcal{E}\mathrm{sp})^{op}_{M'}\times \mathrm{N}(\mathcal{R}\mathrm{ind})^{op}$ is equivalent to the composite map

$$\mathrm{N}(\mathcal{E}\mathrm{sp})^{op}_{M'}\times \mathrm{N}(\mathcal{R}\mathrm{ind})^{op}\xrightarrow{{}_{\mathcal{E}\mathrm{sp}}\mathrm{EO}^*_{\otimes}\mid \mathrm{N}(\mathcal{E}\mathrm{sp})^{op}_{M'}\times \mathrm{N}(\mathcal{R}\mathrm{ind})^{op}} \mathcal{P}\mathrm{r}^{\mathrm{L}\otimes}_{\mathrm{st,cl}}\to \mathcal{C}\mathrm{at}^{\otimes}_{\infty}.$$

In order to compare $^{\text{lis-ét}}_{\mathcal{C}\mathrm{hp}}\mathrm{EO}^*_{\otimes}$ and ${}_{\mathcal{C}\mathrm{hp}}\mathrm{EO}^*_{\otimes}$ more generally, we apply the following variant of Proposition 4.1.1.

Lemma 5.3.1. *Let $(\tilde{\mathcal{C}},\tilde{\mathcal{E}},\tilde{\mathcal{F}})$ be a 2-marked ∞-category such that $\tilde{\mathcal{C}}$ admits pullbacks and $\tilde{\mathcal{E}}\subseteq\tilde{\mathcal{F}}$ are stable under composition and pullback. Let $\mathcal{C}\subseteq\tilde{\mathcal{C}}$ be a full subcategory stable under pullback such that every edge in $\tilde{\mathcal{F}}$ is representable in $\mathcal{C}$ and for every object X of $\tilde{\mathcal{C}}$, there exists a morphism $Y\to X$ in $\tilde{\mathcal{E}}$ with Y in $\mathcal{C}$. Let $\mathcal{D}$ be an ∞-category such that $\mathcal{D}^{op}$ admits geometric realizations. Let $\mathcal{E}=\tilde{\mathcal{E}}\cap\mathcal{C}_1$, $\mathcal{F}=\tilde{\mathcal{F}}\cap\mathcal{C}_1$, and $\mathrm{Fun}^{\mathcal{E}}(\mathcal{C}^{op}_{\mathcal{F}},\mathcal{D})\subseteq \mathrm{Fun}(\mathcal{C}^{op}_{\mathcal{F}},\mathcal{D})$ (resp. $\mathrm{Fun}^{\tilde{\mathcal{E}}}(\tilde{\mathcal{C}}^{op}_{\tilde{\mathcal{F}}},\mathcal{D})\subseteq \mathrm{Fun}(\tilde{\mathcal{C}}^{op}_{\tilde{\mathcal{F}}},\mathcal{D})$) be the full subcategory spanned by functors F such that for every edge $f\colon X_0^+\to X_{-1}^+$ in $\mathcal{E}$ (resp. in $\tilde{\mathcal{E}}$), $F\circ(X_\bullet^+)^{op}$ is a limit diagram $\mathrm{N}(\mathbf{\Delta}^{\mathrm{op}}_{s,+})\to\mathcal{D}$, where $X_\bullet^{s,+}$ is a semisimplicial Čech nerve of f. Then the restriction map*

$$\mathrm{Fun}^{\tilde{\mathcal{E}}}(\tilde{\mathcal{C}}^{op}_{\tilde{\mathcal{F}}},\mathcal{D})\to \mathrm{Fun}^{\mathcal{E}}(\mathcal{C}^{op}_{\mathcal{F}},\mathcal{D})$$

is a trivial fibration.

We will show, by exploiting localized lisse-étale topoi, that $^{\text{lis-ét}}_{\mathcal{C}\mathrm{hp}}\mathrm{EO}^*_{\otimes}$ induces a functor in $\mathrm{Fun}^{\tilde{\mathcal{E}}}(\mathrm{N}(\mathcal{C}\mathrm{hp})^{op}_{M},\mathrm{Fun}(\mathrm{N}(\mathcal{R}\mathrm{ind})^{op},\mathcal{C}\mathrm{at}^{\otimes}_{\infty}))$, where $\tilde{\mathcal{E}}\subseteq M$ is the subset of surjective morphisms. For an object $V\to X$ of $\text{Lis-ét}(X)$, we let $\widetilde{V}$ be the sheaf in $X_{\text{lis-ét}}$ represented by V. The overcategory $(X_{\text{lis-ét}})_{/\widetilde{V}}$ is equivalent to the topos defined by the site $\text{Lis-ét}(X)_{/V}$ endowed with the étale topology [2, III 5.4]. A morphism $f\colon U\to U'$ of $\text{Lis-ét}(X)_{/V}$ induces a 2-commutative diagram

$$\begin{tikzcd}
 & (X_{\text{lis-ét}})_{/\widetilde{U}} \arrow[r, "\epsilon_{U*}"] \arrow[dl, "u_*"'] \arrow[d, "f_*"] & U_{\text{ét}} \arrow[d, "f_{\text{ét}*}"] \\
(X_{\text{lis-ét}})_{/\widetilde{V}} & (X_{\text{lis-ét}})_{/\widetilde{U'}} \arrow[l, "u'_*"] \arrow[r, "\epsilon_{U'*}"] & U'_{\text{ét}}
\end{tikzcd}$$

of topoi [2, IV 5.5]. For $\lambda\in\mathcal{R}\mathrm{ind}$, we let $\mathcal{D}_{\mathrm{cart}}((X_{\text{lis-ét}})_{/\widetilde{V}},\lambda)\subseteq\mathcal{D}((X_{\text{lis-ét}})_{/\widetilde{V}},\lambda)$ be the full subcategory spanned by complexes on which the natural transformation $f^*\circ\epsilon_{U'*}\circ u'^*\to\epsilon_{U*}\circ u^*$ is an isomorphism for all f. The full subcategory is stable under tensor product and contains the monoidal unit, so that it defines a symmetric monoidal ∞-category $\mathcal{D}_{\mathrm{cart}}((\mathcal{X}_{\text{lis-ét}})_{/\widetilde{V}},\lambda)^{\otimes}$.

We have a functor $[1]\times\text{Lis-ét}(X)\times\mathcal{R}\mathrm{ind}\to\mathcal{R}\mathrm{inged}\mathcal{PT}\mathrm{opos}$ sending $[1]\times\{f\colon U\to V\}\times\{(\Xi,\Lambda)\}$ to the square

$$\begin{tikzcd}
((X_{\text{lis-ét}})^{\Xi}_{/\widetilde{U}},\Lambda) \arrow[r, "\epsilon_{U*}"] \arrow[d, "f_*"] & (U^{\Xi}_{\text{ét}},\Lambda) \arrow[d, "f_{\text{ét}*}"] \\
((X_{\text{lis-ét}})^{\Xi}_{/\widetilde{V}},\Lambda) \arrow[r, "\epsilon_{V*}"] & (V^{\Xi}_{\text{ét}},\Lambda).
\end{tikzcd}$$

Composing with the functor $\mathbf{T}^{\otimes}$ (2.1), we obtain a map

$$F\colon(\Delta^1)^{op}\times\mathrm{N}(\text{Lis-ét}(X))^{op}\times\mathrm{N}(\mathcal{R}\mathrm{ind}^{op})\to\mathcal{P}\mathrm{r}^{\mathrm{L}\otimes}_{\mathrm{st,cl}}.$$

By construction, $F([0],V,(\Xi,\Lambda))=\mathcal{D}((\mathcal{X}_{\text{lis-ét}})^{\Xi}_{/\widetilde{V}},\Lambda)^{\otimes}$. Replacing $F([0],V,(\Xi,\Lambda))$ by the full subcategory $\mathcal{D}_{\mathrm{cart}}((\mathcal{X}_{\text{lis-ét}})^{\Xi}_{/\widetilde{V}},\Lambda)$, we obtain a map

$$F'\colon(\Delta^1)^{op}\times\mathrm{N}(\text{Lis-ét}(X))^{op}\times\mathrm{N}(\mathcal{R}\mathrm{ind}^{op})\to\mathcal{C}\mathrm{at}^{\otimes}_{\infty},$$

sending $(\Delta^1)^{op} \times \{f\colon U \to V\} \times \{(\Xi,\Lambda)\}$ to the square

$$\begin{tikzcd}
\mathcal{D}_{\mathrm{cart}}((X_{\text{lis-ét}})^{\Xi}_{/\widetilde{U}},\Lambda)^{\otimes} & \mathcal{D}(U^{\Xi}_{\text{ét}},\Lambda)^{\otimes} \arrow[l, "\epsilon_U^*"'] \\
\mathcal{D}_{\mathrm{cart}}((X_{\text{lis-ét}})^{\Xi}_{/\widetilde{V}},\Lambda)^{\otimes} \arrow[u, "f^*"] & (V^{\Xi}_{\text{ét}},\Lambda)^{\otimes}. \arrow[l, "\epsilon_V^*"'] \arrow[u, "f^*_{\text{ét}}"']
\end{tikzcd}$$

Lemma 5.3.2. *The 1-cell of* $\mathrm{Fun}(\mathrm{N}(\text{Lis-ét}(X))^{op} \times \mathrm{N}(\mathcal{R}\mathrm{ind}^{op}), \mathcal{C}\mathrm{at}_\infty^{\otimes})$ *corresponding to* F' *is an equivalence.*

In particular, the map F' essentially factors through $\mathcal{P}\mathrm{r}^{\mathrm{L}\otimes}_{\mathrm{st,cl}} \subseteq \mathcal{C}\mathrm{at}_\infty^{\otimes}$.

Proof. We only need to prove that for every object V of Lis-ét(X) and every object (Ξ,Λ) of $\mathcal{R}$ind,

$$\epsilon_V^*\colon \mathcal{D}(V^{\Xi}_{\text{ét}},\Lambda) \to \mathcal{D}_{\mathrm{cart}}((X_{\text{lis-ét}})^{\Xi}_{/\widetilde{V}},\Lambda)$$

is an equivalence. Let $\mathbf{R}\epsilon_{V*}$ be a right adjoint of $\epsilon_V^*\colon \mathcal{D}(V^{\Xi}_{\text{ét}},\Lambda) \to \mathcal{D}((X_{\text{lis-ét}})^{\Xi}_{/\widetilde{V}},\Lambda)$, and $\mathbf{R}_{\mathrm{cart}}\epsilon_{V*} = \mathbf{R}\epsilon_{V*} \mid \mathcal{D}_{\mathrm{cart}}((X_{\text{lis-ét}})^{\Xi}_{/\widetilde{V}},\Lambda)$ be the restriction. Then $(\epsilon_V^*, \mathbf{R}_{\mathrm{cart}}\epsilon_{V*})$ is a pair of adjoint functors. We only need to show that the unit transformation $\mathrm{id} \to \mathbf{R}_{\mathrm{cart}}\epsilon_{V*} \circ \epsilon_V^*$ and the counit transformation $\epsilon_V^* \circ \mathbf{R}_{\mathrm{cart}}\epsilon_{V*} \to \mathrm{id}$ are natural equivalences. However, this can be easily checked in the homotopy categories. ◻

Let $v\colon V \to X$ be an object Lis-ét(X), viewed as a morphism in $\mathcal{C}$hp. Assume that v is surjective. Let (Ξ,Λ) be an object of $\mathcal{R}$ind.

Lemma 5.3.3. *A complex* $K \in \mathcal{D}(X^{\Xi}_{\text{lis-ét}},\Lambda)$ *belongs to* $\mathcal{D}_{\mathrm{cart}}(X^{\Xi}_{\text{lis-ét}},\Lambda)$ *if and only if* v^*K *belongs to* $\mathcal{D}_{\mathrm{cart}}((X_{\text{lis-ét}})^{\Xi}_{/\widetilde{V}},\Lambda)$.

Proof. The necessity is trivial. Assume that v^*K belongs to $\mathcal{D}_{\mathrm{cart}}$. We need to show that for every morphism $f\colon Y' \to Y$ of Lis-ét(X), the map $f^*(K \mid Y_{\text{ét}}) \to (K \mid Y'_{\text{ét}})$ is an equivalence. The problem is local for the étale topology on Y. However, locally for the étale topology on Y, $Y \to X$ factors through v [1, 17.16.3 (ii)]. The assertions thus follows from the assumption. ◻

Let $V_\bullet\colon \mathrm{N}(\mathbf{\Delta}_+^{op}) \to \mathrm{N}(\mathcal{C}\mathrm{hp})$ be a Čech nerve of v, which can be viewed as a simplicial object of Lis-ét(X). By Lemma 5.3.3, we can apply Lemma 3.1.3 to $U_\bullet = \widetilde{V_\bullet}^{\Xi}$ and $\mathcal{C}_\bullet = \mathrm{Mod}_{\mathrm{cart}}((X_{\text{lis-ét}})^{\Xi}_{\widetilde{V_\bullet}},\Lambda)$. We obtain a natural equivalence of symmetric monoidal ∞-categories

$$\mathcal{D}_{\mathrm{cart}}(X^{\Xi}_{\text{lis-ét}},\Lambda)^{\otimes} \xrightarrow{\sim} \varprojlim_{n\in\mathbf{\Delta}} \mathcal{D}_{\mathrm{cart}}((X_{\text{lis-ét}})^{\Xi}_{/\widetilde{V_n}},\Lambda)^{\otimes},$$

functorial in (Ξ,Λ). Combining this with a quasi-inverse of the equivalence in Lemma 5.3.2, we obtain the following result.

Proposition 5.3.4 (Cohomological descent for lisse-étale topoi)**.** *Let X be an Artin stack, V be an algebraic space, and $v\colon V \to X$ be a surjective smooth morphism. Then there is an equivalence in* $\mathrm{Fun}(\mathrm{N}(\mathcal{R}\mathrm{ind}^{op}), \mathcal{P}\mathrm{r}^{\mathrm{L}\otimes}_{\mathrm{st,cl}})$ *sending* (Ξ,Λ) *to*

$$\mathcal{D}_{\mathrm{cart}}(X^{\Xi}_{\text{lis-ét}},\Lambda)^{\otimes} \xrightarrow{\sim} \varprojlim_{n\in\mathbf{\Delta}} \mathcal{D}(V^{\Xi}_{n,\text{ét}},\Lambda)^{\otimes},$$

where $V_\bullet$ is a Čech nerve of v.

By construction, the above equivalence is compatible with pullback by smooth representable morphisms to X. Therefore, it implies the following.

Corollary 5.3.5. *Let $f\colon Y \to X$ be a smooth surjective representable morphism of Artin stacks, $Y_\bullet$ be a Čech nerve of f. Then the map* $\mathcal{D}_{\mathrm{cart}}(X^{\Xi}_{\text{lis-ét}},\Lambda)^{\otimes} \xrightarrow{\sim} \varprojlim_{n\in\mathbf{\Delta}_s} \mathcal{D}(Y^{\Xi}_{n,\text{lis-ét}},\Lambda)^{\otimes}$ *is an equivalence.*

In other words, ${}^{\text{lis-ét}}_{\mathcal{C}\mathrm{hp}}\mathrm{EO}^*_{\otimes}$ induces a functor in $\mathrm{Fun}^{\tilde{\mathcal{E}}}(\mathrm{N}(\mathcal{C}\mathrm{hp})^{op}_M, \mathrm{Fun}(\mathrm{N}(\mathcal{R}\mathrm{ind})^{op}, \mathcal{C}\mathrm{at}^{\otimes}_{\infty}))$. Applying 5.3.1, we obtain the following.

Corollary 5.3.6. *The map* ${}^{\text{lis-ét}}_{\mathcal{C}\mathrm{hp}}\mathrm{EO}^*_{\otimes}$ *is equivalent to the composite map*

$$\mathrm{N}(\mathcal{C}\mathrm{hp})^{op}_M \times \mathrm{N}(\mathcal{R}\mathrm{ind})^{op} \xrightarrow{{}_{\mathcal{C}\mathrm{hp}}\mathrm{EO}^*_{\otimes}|\mathrm{N}(\mathcal{C}\mathrm{hp})^{op}_M\times\mathrm{N}(\mathcal{R}\mathrm{ind})^{op}} \mathcal{P}\mathrm{r}^{\mathrm{L}\otimes}_{\mathrm{st,cl}} \to \mathcal{C}\mathrm{at}^{\otimes}_{\infty}.$$

In particular, for every Artin stack X and every object (Ξ,Λ) of $\mathcal{R}\mathrm{ind}$, we have an equivalence $\mathcal{D}_{\mathrm{cart}}(X^{\Xi}_{\text{lis-ét}},\Lambda)^{\otimes} \simeq \mathcal{D}(X,(\Xi,\Lambda))^{\otimes}$, and consequently $\mathcal{D}_{\mathrm{cart}}(X^{\Xi}_{\text{lis-ét}},\Lambda)^{\otimes}$ is a closed presentable stable symmetric monoidal ∞-category.

Corollary 5.3.7. *Let X be an Artin stack, (Ξ,Λ) be an object of $\mathcal{R}\mathrm{ind}$. Under the above equivalence, the usual t-structure on $\mathcal{D}_{\mathrm{cart}}(X^{\Xi}_{\text{lis-ét}},\Lambda)$ corresponds to the t-structure on $\mathcal{D}(X,(\Xi,\Lambda))$ obtained in Output II. In particular, the heart of $\mathcal{D}(X,(\Xi,\Lambda))$ is equivalent to (the nerve of) the abelian category of Cartesian $(X^{\Xi}_{\text{lis-ét}},\Lambda)$-modules.*

Remark 5.3.8 (de Jong). The $*$-pullback encoded by ${}_{\mathcal{C}\mathrm{hp}}\mathrm{EO}^*_{\otimes}$ can be described more directly using big étale topoi of Artin stacks. Let $\mathcal{W}$ be a universe such that $\mathcal{V}$ belongs to $\mathcal{W}$. We add a hat to a notation to denote the version relative to the bigger universes $\mathcal{V}$ and $\mathcal{W}$, instead of $\mathcal{U}$ and $\mathcal{V}$. For example, we denote by $\widehat{\mathcal{R}\mathrm{inged}\mathcal{P}\mathcal{T}\mathrm{opos}}$ the $(2,1)$-category of $\mathcal{V}$-topoi belonging to $\mathcal{W}$ with enough points (cf. Notation 2.2.4). For any Artin stack X, we consider the full subcategories $\mathcal{E}\mathrm{sp}_{/X} \subseteq \mathcal{C}\mathrm{hp}_{\mathrm{rep}/X}$ of $\mathcal{C}\mathrm{hp}_{/X}$ spanned by morphisms whose sources are algebraic spaces and representable morphisms, respectively. They are ordinary categories and we endow them with the étale topology. The corresponding $\mathcal{V}$-topos, namely the categories of $\mathcal{V}$-sheaves on these sites, are equivalent, and we denote them by $X_{\text{big.ét}}$. The construction of $\mathcal{C}\mathrm{hp}_{\mathrm{rep}/X}$ is functorial in X, so that we obtain a functor $\mathcal{C}\mathrm{hp}\times\mathcal{R}\mathrm{ind} \to \widehat{\mathcal{R}\mathrm{inged}\mathcal{P}\mathcal{T}\mathrm{opos}}$. Composing with $\widehat{\mathbf{T}^{\otimes}}$, we obtain a functor $\mathrm{N}(\mathcal{C}\mathrm{hp})^{op}\times\mathrm{N}(\mathcal{R}\mathrm{ind})^{op} \to \widehat{\mathcal{P}\mathrm{r}^{\mathrm{L}\otimes}_{\mathrm{st,cl}}}$ sending $(X,(\Xi,\Lambda))$ to $\mathcal{D}(X^{\Xi}_{\text{big.ét}},\Lambda)^{\otimes}$. Replacing the latter by the full subcategory $\mathcal{D}_{\mathrm{cart}}(X^{\Xi}_{\text{big.ét}},\Lambda)^{\otimes}$ consisting of complexes K such that $f^*(K \mid Y'_{\text{ét}}) \to (K \mid Y_{\text{ét}})$ is an equivalence for every morphism $f\colon Y\to Y'$ of $\mathcal{E}\mathrm{sp}_{/X}$, we obtain a functor

$${}^{\mathrm{big}}_{\mathcal{C}\mathrm{hp}}\mathrm{EO}^*_{\otimes}\colon \mathrm{N}(\mathcal{C}\mathrm{hp})^{op}\times\mathrm{N}(\mathcal{R}\mathrm{ind})^{op} \to \widehat{\mathcal{C}\mathrm{at}^{\otimes}_{\infty}}.$$

Using similar arguments as in this section, with Lemma 5.3.1 replaced by Proposition 4.1.1, one shows that ${}^{\mathrm{big}}_{\mathcal{C}\mathrm{hp}}\mathrm{EO}^*_{\otimes}$ is equivalent to the composition

$$\mathrm{N}(\mathcal{C}\mathrm{hp})^{op}\times\mathrm{N}(\mathcal{R}\mathrm{ind})^{op} \xrightarrow{{}_{\mathcal{C}\mathrm{hp}}\mathrm{EO}^*_{\otimes}} \mathcal{P}\mathrm{r}^{\mathrm{L}\otimes}_{\mathrm{st,cl}} \to \widehat{\mathcal{C}\mathrm{at}^{\otimes}_{\infty}}.$$

5.4. **Higher Artin stacks.** We begin by recalling the definition of higher Artin stacks. We will use the fppf topology instead of the étale topology adopted in [37]. The two definitions are equivalent [38]. Let $\mathcal{S}\mathrm{ch}^{\mathrm{aff}} \subseteq \mathcal{S}\mathrm{ch}$ be the full subcategory spanned by affine schemes. Recall that $\mathcal{S}_{\mathcal{W}}$ is the ∞-category of spaces in $\mathcal{W}\in\{\mathcal{U},\mathcal{V}\}$[7].

Definition 5.4.1 (Prestack and stack). We defined the ∞-category of *($\mathcal{V}$-)prestacks* to be $\mathcal{C}\mathrm{hp}^{\mathrm{pre}} = \mathrm{Fun}(\mathrm{N}(\mathcal{S}\mathrm{ch}^{\mathrm{aff}})^{op},\mathcal{S}_{\mathcal{V}})$. We endow $\mathrm{N}(\mathcal{S}\mathrm{ch}^{\mathrm{aff}})$ with the fppf topology. We define the ∞-category of (small) stacks $\mathcal{C}\mathrm{hp}^{\mathrm{fppf}}$ to be the essential image of the following inclusion

$$\mathrm{Shv}(\mathrm{N}(\mathcal{S}\mathrm{ch}^{\mathrm{aff}})_{\mathrm{fppf}}) \cap \mathrm{Fun}(\mathrm{N}(\mathcal{S}\mathrm{ch}^{\mathrm{aff}})^{op},\mathcal{S}_{\mathcal{U}}) \subseteq \mathcal{C}\mathrm{hp}^{\mathrm{pre}},$$

where $\mathrm{Shv}(\mathrm{N}(\mathcal{S}\mathrm{ch}^{\mathrm{aff}})_{\mathrm{fppf}}) \subseteq \mathrm{Fun}(\mathrm{N}(\mathcal{S}\mathrm{ch}^{\mathrm{aff}})^{op},\mathcal{S}_{\mathcal{V}})$ is the full subcategory spanned by fppf sheaves [29, 6.2.2.6]. A prestack F is *k-truncated* [29, 5.5.6.1] for an integer $n\geq -1$, if $\pi_i(F(A))=0$ for every object A of $\mathcal{S}\mathrm{ch}^{\mathrm{aff}}$ and every integer $i>k$.

The Yoneda embedding $\mathrm{N}(\mathcal{S}\mathrm{ch}^{\mathrm{aff}}) \to \mathcal{C}\mathrm{hp}^{\mathrm{pre}}$ extends to a fully faithful functor $\mathrm{N}(\mathcal{E}\mathrm{sp}) \to \mathcal{C}\mathrm{hp}^{\mathrm{pre}}$ sending X to the discrete Kan complex $\mathrm{Hom}_{\mathcal{E}\mathrm{sp}}(\mathrm{Spec}\,A, X)$. The image of this functor is contained in $\mathcal{C}\mathrm{hp}^{\mathrm{fppf}}$. We will generally not distinguish between $\mathrm{N}(\mathcal{E}\mathrm{sp})$ and its essential image in $\mathcal{C}\mathrm{hp}^{\mathrm{fppf}}$. A stack X belongs to the latter if and only if it satisfies the following conditions.

[7] We refer to §0.5 for conventions on set-theoretical issues.

- It is 0-truncated.
- The diagonal morphism $X \to X \times X$ is schematic, that is, for every morphism $Z \to X \times X$ with Z a scheme, the fiber product $X \times_{X \times X} Z$ is a scheme.
- There exists a scheme Y and a morphism $f\colon Y \to X$ that is (automatically schematic,) smooth (resp. étale) and surjective. In other words, for every morphism $Z \to X$ with Z a scheme, the induced morphism $Y \times_X Z \to Z$ is smooth (resp. étale) and surjective. The morphism f is called an *atlas* (resp. *étale atlas*) for X.

Definition 5.4.2 (Higher Artin stack; see [12, 37]). We define k-Artin stacks inductively for $k \geq 0$.

- A stack X is a 0-*Artin stack* if it belongs to the essential image of $\mathrm{N}(\mathcal{E}\mathrm{sp})$.

For $k \geq 0$, assume that we have defined k-Artin stacks. We define:

- A morphism $F' \to F$ of prestacks is k-*Artin* if for every morphism $Z \to F$ where Z is a k-Artin stack, the fiber product $F' \times_F Z$ is a k-Artin stack.
- A k-Artin morphism $F' \to F$ is flat (resp. locally of finite type, resp. locally of finite presentation, resp. smooth, resp. surjective) if for every morphism $Z \to F$ and every atlas $f\colon Y \to F' \times_F Z$ where Y and Z are schemes, the composed morphism $Y \to F' \times_F Z \to Z$ is a flat (resp. locally of finite type, resp. locally of finite presentation, resp. smooth, resp. surjective) morphism of schemes.
- A stack X is a $(k+1)$-Artin stack if the diagonal morphism $X \to X \times X$ is k-Artin, and there exists a scheme Y together with a morphism $f\colon Y \to X$ that is (automatically k-Artin,) smooth and surjective. The morphism f is called an *atlas* for X.

We denote by $\mathcal{C}\mathrm{hp}^{k\text{-Ar}} \subseteq \mathcal{C}\mathrm{hp}^{\mathrm{fppf}}$ the full subcategory spanned by k-Artin stacks. We define *higher Artin stacks* to be objects of $\mathcal{C}\mathrm{hp}^{\mathrm{Ar}} = \bigcup_{k \geq 0} \mathcal{C}\mathrm{hp}^{k\text{-Ar}}$. A morphism $F' \to F$ of prestacks is *higher Artin* if for every morphism $Z \to F$ where Z is a higher Artin stack, the fiber product $F' \times_F Z$ is a higher Artin stack.

To simplify notations, we let $\mathcal{C}\mathrm{hp}^{(-1)\text{-Ar}} = \mathrm{N}(\mathcal{S}\mathrm{ch}^{\mathrm{qs}})$ and $\mathcal{C}\mathrm{hp}^{(-2)\text{-Ar}} = \mathrm{N}(\mathcal{S}\mathrm{ch}^{\mathrm{qc.sep}})$, and we call their objects (-1)-Artin stacks and (-2)-Artin stacks, respectively.

By definition, $\mathcal{C}\mathrm{hp}^{0\text{-Ar}}$ and $\mathcal{C}\mathrm{hp}^{1\text{-Ar}}$ are equivalent to $\mathrm{N}(\mathcal{E}\mathrm{sp})$ and $\mathrm{N}(\mathcal{C}\mathrm{hp})$, respectively. For $k \geq 0$, k-Artin stacks are k-truncated prestacks. Higher Artin stacks are *hypercomplete* sheaves [29, 6.5.2.9]. Every flat surjective morphism locally of finite presentation of higher Artin stacks is an *effective epimorphism* in the ∞-topos $\mathrm{Shv}(\mathrm{N}(\mathcal{S}\mathrm{ch}^{\mathrm{aff}})_{\mathrm{fppf}})$ in the sense after [29, 6.2.3.5]. A higher Artin morphism of prestacks is k-Artin for some $k \geq 0$.

Definition 5.4.3.

- A higher Artin stack X is *quasi-compact* if there exists an atlas $f\colon Y \to X$ such that Y is a quasi-compact scheme.
- A higher Artin morphism $F' \to F$ of prestacks is *quasi-compact* if for every morphism $Z \to F$ where Z is a quasi-compact scheme, the fiber product $F' \times_F Z$ is a quasi-compact higher Artin stack.

We define quasi-separated higher Artin morphisms of prestacks by induction as follows.

- A 0-Artin morphism of prestacks $F' \to F$ is *quasi-separated* if the diagonal morphism $F' \to F' \times_F F'$, which is automatically schematic, is quasi-compact.
- For $k \geq 0$, a $(k+1)$-Artin morphism of prestacks $F' \to F$ is *quasi-separated* if the diagonal morphism $F' \to F' \times_F F'$, which is automatically k-Artin, is quasi-separated and quasi-compact.

We say a higher Artin stack X is L-coprime if there exists a morphism $X \to \operatorname{Spec} \mathbb{Z}[\mathsf{L}^{-1}]$. This is equivalent to the existence of an L-coprime atlas. We denote by $\mathcal{C}\mathrm{hp}^{\mathrm{Ar}}_{\mathsf{L}} \subseteq \mathcal{C}\mathrm{hp}^{\mathrm{Ar}}$ the full subcategory spanned by L-coprime higher Artin stacks. We let $\mathcal{C}\mathrm{hp}^{k\text{-Ar}}_{\mathsf{L}} = \mathcal{C}\mathrm{hp}^{k\text{-Ar}} \cap \mathcal{C}\mathrm{hp}^{\mathrm{Ar}}_{\mathsf{L}}$.

Definition 5.4.4 (Relative dimension). We define by induction the class of smooth morphisms *of pure relative dimension* d of k-Artin stacks for $d \in \mathbb{Z} \cup \{-\infty\}$ and the *upper relative dimension* $\dim^+(f)$ for every morphism f locally of finite type of k-Artin stacks. If in Input 0 of §4.1, we let $\tilde{\mathcal{F}}$ (resp. $\tilde{\mathcal{E}}''$, $\tilde{\mathcal{E}}''_d$) be

the set of morphisms locally of finite type (resp. smooth morphisms, smooth morphisms of pure relative dimension d) of k-Artin stacks, then such definitions should satisfy conditions (6) through (9) of Input 0.

When $k=-1$, we use the usual definitions for classical schemes, with the upper relative dimension given in Definition 4.1.9. For $k\geq -1$, assuming that these notions are defined for k-Artin stacks. We first extend these definitions to k-representable morphisms locally of finite type of $(k+1)$-Artin stacks. Let $f\colon Y\to X$ be such a morphism, and $X_0\xrightarrow{u} X$ be an atlas of X. Let $f_0\colon Y_0\to X_0$ be the base change of f by u. Then f_0 is a morphism locally of finite type of k-Artin stacks. We define $\dim^+(f)=\dim^+(f_0)$. It is easy to see that this is independent of the atlas we choose, by assumption (9d) of Input 0. We say f is smooth of pure relative dimension d if f_0 is. This is independent of the atlas we choose by assumption (7) of Input 0. We need to check (6) through (9) of Input 0. (7) through (9) are easy and (6) can be argued as follows. Since f_0 is a smooth morphism of k-Artin stacks, there is a decomposition $f_0\colon Y_0\simeq\coprod_{d\in\mathbb{Z}}Y_{0,d}\xrightarrow{(f_{0,d})}X_0$. Let $X_\bullet\to X$ be a Čech nerve of u, and $Y_{\bullet,d}=Y_{0,d}\times_{X_0}X_\bullet$. Then $\coprod_{d\in\mathbb{Z}}Y_{\bullet,d}\to Y$ is a Čech nerve of $v\colon Y_0\to Y$. Let $Y_d=\varinjlim_{n\in\mathbf{\Delta}^{op}}Y_{n,d}$. Then $Y\simeq\coprod_{d\in\mathbb{Z}}Y_d$ is the desired decomposition.

Next we extend these definitions to all morphisms locally of finite type of $(k+1)$-Artin stacks. Let $f\colon Y\to X$ be such a morphism, and $v_0\colon Y_0=\coprod_{d\in\mathbb{Z}}Y_{0,d}\xrightarrow{(v_{0,d})}Y$ be an atlas of Y such that $v_{0,d}$ is smooth of pure relative dimension d. We define $\dim^+(f)=\sup_{d\in\mathbb{Z}}\{\dim^+(f\circ v_{0,d})-d\}$. We say f is smooth of pure relative dimension d if for every $e\in\mathbb{Z}$, $f\circ v_{0,e}$ is smooth of pure relative dimension $d+e$. We leave it to the reader to check that these definitions are independent of the atlas we choose, and satisfy (7) through (9) of Input 0. We sketch the proof for (6). Since $f\circ v_{0,e}$ is smooth and k-representable, it can be decomposed as $Y_{0,e}\simeq\coprod_{e'\in\mathbb{Z}}Y_{0,e,e'}\xrightarrow{(f_{e,e'})}X$ such that $f_{e,e'}$ is of pure relative dimension e'. We let Y_d be the colimit of the underlying groupoid object of the Čech nerve of $\coprod_{e'-e=d}Y_{0,e,e'}\to X$. Then $Y\simeq\coprod_{d\in\mathbb{Z}}Y_d\to X$ is the desired decomposition.

It is clear that the definition of the upper relative dimension for $k=0$ coincides with Definition 4.1.9.

Let A be the set of all morphisms of higher Artin stacks, and $F\subseteq A$ be the set of morphisms locally of finite type. For every $k\geq 0$, we are going to construct two maps

$$
\begin{gathered}
{}_{\mathcal{C}\mathrm{hp}_{\mathsf{L}}^{k\text{-}\mathrm{Ar}}}\mathrm{EO}\colon \delta^*_{2,\{2\}}\mathrm{Fun}(\Delta^1,\mathcal{C}\mathrm{hp}_{\mathsf{L}}^{k\text{-}\mathrm{Ar}})^{\mathrm{cart}}_{F^0,A\to}\to\mathrm{Fun}(\mathrm{N}(\mathcal{R}\mathrm{ind}^{op}_{\mathsf{L}\text{-}\mathrm{tor}}),\mathrm{Mon}^{\mathcal{P}\mathrm{r}^{\mathrm{L}}_{\mathrm{st}}}_{\mathcal{P}\mathrm{f}}(\mathcal{C}\mathrm{at}_\infty));\\
{}_{\mathcal{C}\mathrm{hp}^{k\text{-}\mathrm{Ar}}}\mathrm{EO}^*_\otimes\colon(\mathcal{C}\mathrm{hp}^{k\text{-}\mathrm{Ar}})^{op}\to\mathrm{Fun}(\mathrm{N}(\mathcal{R}\mathrm{ind}^{op}),\mathcal{P}\mathrm{r}^{\mathrm{L}\otimes}_{\mathrm{st,cl}}),
\end{gathered}
$$

such that their restrictions to $(k-1)$-Artin stacks coincide with those for the latter.

We construct by induction. When $k=-2,-1,0,1$, they have been constructed in §§3.2, 5.1, 5.2, 5.3, respectively. Assuming that they have been extended to k-Artin stacks. We run the *simplified* DESCENT with the following input:

- $\tilde{\mathcal{C}}=\mathcal{C}\mathrm{hp}^{(k+1)\text{-}\mathrm{Ar}}$. It is geometric.
- $\mathcal{C}=\mathcal{C}\mathrm{hp}^{k\text{-}\mathrm{Ar}}$, $\mathsf{s}''\to\mathsf{s}'$ is the identity morphism of $\operatorname{Spec}\mathbb{Z}[\mathsf{L}^{-1}]$. In particular, $\mathcal{C}'=\mathcal{C}''=\mathcal{C}\mathrm{hp}^{k\text{-}\mathrm{Ar}}_{\mathsf{L}}$, and $\tilde{\mathcal{C}}'=\tilde{\mathcal{C}}''=\mathcal{C}\mathrm{hp}^{(k+1)\text{-}\mathrm{Ar}}_{\mathsf{L}}$.
- $\tilde{\mathcal{E}}_{\mathrm{s}}$ is the set of *surjective* morphisms of $(k+1)$-Artin stacks.
- $\tilde{\mathcal{E}}'=\tilde{\mathcal{E}}''$ is the set of *smooth* morphisms of $(k+1)$-Artin stacks.
- $\tilde{\mathcal{E}}''_d$ is the set of *smooth* morphisms of $(k+1)$-Artin stacks of pure relative dimension d.
- $\tilde{\mathcal{E}}_{\mathrm{t}}$ is the set of *flat* morphisms *locally of finite presentation* of $(k+1)$-Artin stacks.
- $\tilde{\mathcal{F}}=F$ is the set of morphisms *locally of finite type* of $(k+1)$-Artin stacks.
- $\mathcal{L}=\mathrm{N}(\mathcal{R}\mathrm{ind}^{op})$, and $\mathcal{L}'=\mathcal{L}''=\mathrm{N}(\mathcal{R}\mathrm{ind}^{op}_{\mathsf{L}\text{-}\mathrm{tor}})$.
- $\dim^+$ is the upper relative dimension in Definition 5.4.4.
- Input I and II is given by induction hypothesis. In particular, ${}_{\mathcal{C}}\mathrm{EO}={}_{\mathcal{C}\mathrm{hp}_{\mathsf{L}}^{k\text{-}\mathrm{Ar}}}\mathrm{EO}$ and ${}_{\mathcal{C}}\mathrm{EO}^*_\otimes={}_{\mathcal{C}\mathrm{hp}^{k\text{-}\mathrm{Ar}}}\mathrm{EO}^*_\otimes$.

Then the output consists of two maps ${}_{\mathcal{C}\mathrm{hp}_{\mathsf{L}}^{(k+1)\text{-}\mathrm{Ar}}}\mathrm{EO}$, ${}_{\mathcal{C}\mathrm{hp}^{(k+1)\text{-}\mathrm{Ar}}}\mathrm{EO}^*_{\otimes}$ and Output II. Taking union of all $k \geq 0$, we obtain the following two maps

$$
\begin{aligned}
{}_{\mathcal{C}\mathrm{hp}_{\mathsf{L}}^{\mathrm{Ar}}}\mathrm{EO}\colon \delta^*_{2,\{2\}}\mathrm{Fun}(\Delta^1, \mathcal{C}\mathrm{hp}_{\mathsf{L}}^{\mathrm{Ar}})^{\mathrm{cart}}_{F^0,A\rightarrow} &\to \mathrm{Fun}(\mathrm{N}(\mathcal{R}\mathrm{ind}^{op}_{\mathsf{L}\text{-tor}}), \mathrm{Mon}^{\mathcal{P}\mathrm{r}^{\mathrm{L}}_{\mathrm{st}}}_{\mathcal{P}\mathrm{f}}(\mathcal{C}\mathrm{at}_\infty));\\
{}_{\mathcal{C}\mathrm{hp}^{\mathrm{Ar}}}\mathrm{EO}^*_{\otimes}\colon (\mathcal{C}\mathrm{hp}^{\mathrm{Ar}})^{op} &\to \mathrm{Fun}(\mathrm{N}(\mathcal{R}\mathrm{ind}^{op}), \mathcal{P}\mathrm{r}^{\mathrm{L}\otimes}_{\mathrm{st,cl}}).
\end{aligned}
$$

Remark 5.4.5. In fact, ${}_{\mathcal{C}\mathrm{hp}^{\mathrm{Ar}}}\mathrm{EO}^*_{\otimes}$ is just a right Kan extension of ${}_{\mathcal{S}\mathrm{ch}}\mathrm{EO}^*_{\otimes}$ along the full inclusion $\mathrm{N}(\mathcal{S}\mathrm{ch}) \subseteq \mathcal{C}\mathrm{hp}^{\mathrm{Ar}}$ of ∞-categories.

5.5. **Higher Deligne–Mumford stacks.** The definition of higher Deligne–Mumford (DM) stacks is similar to that of higher Artin stacks (Definition 5.4.2).

Definition 5.5.1 (Higher DM stack).

- A stack X is a 0-*DM stack* if it belongs to the essential image of $\mathrm{N}(\mathcal{E}\mathrm{sp})$.

For $k \geq 0$, assume that we have defined k-DM stacks. We define:

- A morphism $F' \to F$ of prestacks is k-*DM* if for every morphism $Z \to F$ where Z is a k-DM stack, the fiber product $F' \times_F Z$ is a k-DM stack.
- A k-DM morphism $F' \to F$ of prestacks is *étale* (resp. *locally quasi-finite*) if for every morphism $Z \to F$ and every étale atlas $f\colon Y \to F' \times_F Z$ where Y and Z are schemes, the composed morphism $Y \to F' \times_F Z \to Z$ is an étale (resp. locally quasi-finite) morphism of schemes.
- A stack X is a $(k+1)$-DM stack if the diagonal morphism $X \to X \times X$ is k-DM, and there exists a scheme Y together with a morphism $f\colon Y \to X$ that is (automatically k-DM,) étale and surjective. The morphism f is called an *étale atlas* for X.

We denote by $\mathcal{C}\mathrm{hp}^{k\text{-DM}} \subseteq \mathcal{C}\mathrm{hp}^{\mathrm{fppf}}$ the full subcategory spanned by k-DM stacks. We define *higher DM stacks* to be objects of $\mathcal{C}\mathrm{hp}^{\mathrm{DM}} = \bigcup_{k\geq 0} \mathcal{C}\mathrm{hp}^{k\text{-DM}}$. We put $\mathcal{C}\mathrm{hp}_{\mathsf{L}}^{\mathrm{DM}} = \mathcal{C}\mathrm{hp}^{\mathrm{DM}} \cap \mathcal{C}\mathrm{hp}_{\mathsf{L}}^{\mathrm{Ar}}$, $\mathcal{C}\mathrm{hp}_{\mathsf{L}}^{k\text{-DM}} = \mathcal{C}\mathrm{hp}^{k\text{-DM}} \cap \mathcal{C}\mathrm{hp}_{\mathsf{L}}^{\mathrm{DM}}$,

A morphism of higher DM stacks is étale if and only if it is smooth of pure dimension 0.

Let A be the set of all morphisms of higher DM stacks, and $F \subseteq A$ be the set of morphisms locally of finite type. For every $k \geq 0$, we are going to construct two maps

$$
\begin{aligned}
{}_{\mathcal{C}\mathrm{hp}^{k\text{-DM}}}\mathrm{EO}\colon \delta^*_{2,\{2\}}\mathrm{Fun}(\Delta^1, \mathcal{C}\mathrm{hp}^{k\text{-DM}})^{\mathrm{cart}}_{F^0,A\rightarrow} &\to \mathrm{Fun}(\mathrm{N}(\mathcal{R}\mathrm{ind}^{op}_{\mathrm{tor}}), \mathrm{Mon}^{\mathcal{P}\mathrm{r}^{\mathrm{L}}_{\mathrm{st}}}_{\mathcal{P}\mathrm{f}}(\mathcal{C}\mathrm{at}_\infty));\\
{}_{\mathcal{C}\mathrm{hp}^{k\text{-DM}}}\mathrm{EO}^*_{\otimes}\colon (\mathcal{C}\mathrm{hp}^{k\text{-DM}})^{op} &\to \mathrm{Fun}(\mathrm{N}(\mathcal{R}\mathrm{ind}^{op}), \mathcal{P}\mathrm{r}^{\mathrm{L}\otimes}_{\mathrm{st,cl}}),
\end{aligned}
$$

such that their restrictions to $(k-1)$-DM stacks coincide with those for the latter. The second map has already been constructed in §5.4, after restriction. However for induction, we construct it again, which in fact coincides with the previous one.

We construct by induction. When $k = 0$, they have been constructed in §5.2. Assuming that they have been extended to k-DM stacks. We run the program DESCENT with the following input:

- $\tilde{\mathcal{C}} = \mathcal{C}\mathrm{hp}^{(k+1)\text{-DM}}$. It is geometric.
- $\mathcal{C} = \mathcal{C}\mathrm{hp}^{k\text{-DM}}$, $\mathsf{s}'' \to \mathsf{s}'$ is the morphism $\mathrm{Spec}\,\mathbb{Z}[\mathsf{L}^{-1}] \to \mathrm{Spec}\,\mathbb{Z}$.
- $\tilde{\mathcal{E}}_{\mathrm{s}}$ is the set of *surjective* morphisms of $(k+1)$-DM stacks.
- $\tilde{\mathcal{E}}'$ is the set of *étale* morphisms of $(k+1)$-DM stacks.
- $\tilde{\mathcal{E}}''$ is the set of *smooth* morphisms of $(k+1)$-DM stacks.
- $\tilde{\mathcal{E}}''_d$ is the set of *smooth* morphisms of $(k+1)$-DM stacks of pure relative dimension d.
- $\tilde{\mathcal{E}}_{\mathrm{t}}$ is the set of *flat* morphisms *locally of finite presentation* of $(k+1)$-DM stacks.
- $\tilde{\mathcal{F}} = F$ is the set of morphisms *locally of finite type* of $(k+1)$-DM stacks.
- $\mathcal{L} = \mathrm{N}(\mathcal{R}\mathrm{ind}^{op})$, $\mathcal{L}' = \mathrm{N}(\mathcal{R}\mathrm{ind}^{op}_{\mathrm{tor}})$, and $\mathcal{L}'' = \mathrm{N}(\mathcal{R}\mathrm{ind}^{op}_{\mathsf{L}\text{-tor}})$.
- $\dim^+$ is the upper relative dimension.
- Input I and II is given by induction hypothesis. In particular, ${}_{\mathcal{C}'}\mathrm{EO} = {}_{\mathcal{C}\mathrm{hp}^{k\text{-DM}}}\mathrm{EO}$, and ${}_{\mathcal{C}}\mathrm{EO}^*_{\otimes} = {}_{\mathcal{C}\mathrm{hp}^{k\text{-DM}}}\mathrm{EO}^*_{\otimes}$.

Then the output consists of two maps ${}_{\mathcal{C}\mathrm{hp}^{(k+1)\text{-}\mathrm{DM}}}\mathrm{EO}$, ${}_{\mathcal{C}\mathrm{hp}^{(k+1)\text{-}\mathrm{DM}}}\mathrm{EO}^*_{\otimes}$ and Output II. Taking union of all $k \geq 0$, we obtain the following two maps

$$ {}_{\mathcal{C}\mathrm{hp}^{\mathrm{DM}}}\mathrm{EO}\colon \delta^*_{2,\{2\}}\mathrm{Fun}(\Delta^1, \mathcal{C}\mathrm{hp}^{\mathrm{DM}})^{\mathrm{cart}}_{F^0,A\rightarrow} \to \mathrm{Fun}(\mathrm{N}(\mathcal{R}\mathrm{ind}^{op}_{\mathrm{tor}}), \mathrm{Mon}^{\mathcal{P}\mathrm{r}^{\mathrm{L}}_{\mathrm{st}}}_{\mathcal{P}\mathrm{f}}(\mathcal{C}\mathrm{at}_\infty)); $$
$$ {}_{\mathcal{C}\mathrm{hp}^{\mathrm{DM}}}\mathrm{EO}^*_{\otimes}\colon (\mathcal{C}\mathrm{hp}^{\mathrm{DM}})^{op} \to \mathrm{Fun}(\mathrm{N}(\mathcal{R}\mathrm{ind}^{op}), \mathcal{P}\mathrm{r}^{\mathrm{L}\otimes}_{\mathrm{st,cl}}). $$

Remark 5.5.2. We have the following compatibility:

- The restriction ${}_{\mathcal{C}\mathrm{hp}^{\mathrm{Ar}}}\mathrm{EO}^*_{\otimes} \mid (\mathcal{C}\mathrm{hp}^{\mathrm{DM}})^{op}$ is equivalent to ${}_{\mathcal{C}\mathrm{hp}^{\mathrm{DM}}}\mathrm{EO}^*_{\otimes}$.
- The restriction of ${}_{\mathcal{C}\mathrm{hp}^{\mathrm{DM}}}\mathrm{EO}$ to $\mathcal{C}\mathrm{hp}^{\mathrm{DM}}_{\mathsf{L}}$ and $\mathcal{R}\mathrm{ind}_{\mathsf{L}\text{-tor}}$ is equivalent to the restriction of ${}_{\mathcal{C}\mathrm{hp}^{\mathrm{Ar}}_{\mathsf{L}}}\mathrm{EO}$ to $\mathcal{C}\mathrm{hp}^{\mathrm{DM}}_{\mathsf{L}}$.

Variant 5.5.3. We denote by $Q \subseteq F$ the set of locally quasi-finite morphisms. Applying DESCENT to the map ${}_{\mathcal{S}\mathrm{ch}^{\mathrm{qc.sep}}}^{\mathrm{lqf}}\mathrm{EO}$ constructed in Variant 3.2.6, we obtain

$$ {}_{\mathcal{C}\mathrm{hp}^{\mathrm{DM}}}^{\mathrm{lqf}}\mathrm{EO}\colon \delta^*_{2,\{2\}}\mathrm{Fun}(\Delta^1, \mathcal{C}\mathrm{hp}^{\mathrm{DM}})^{\mathrm{cart}}_{Q^0,A\rightarrow} \to \mathrm{Fun}(\mathrm{N}(\mathcal{R}\mathrm{ind}^{op}), \mathrm{Mon}^{\mathcal{P}\mathrm{r}^{\mathrm{L}}_{\mathrm{st}}}_{\mathcal{P}\mathrm{f}}(\mathcal{C}\mathrm{at}_\infty)). $$

This map and ${}_{\mathcal{C}\mathrm{hp}^{\mathrm{DM}}}\mathrm{EO}$ are equivalent when restricted to their common domain.

Remark 5.5.4. The ∞-category $\mathcal{C}\mathrm{hp}^{\mathrm{DM}}$ can be identified with a full subcategory of the ∞-category $\mathrm{Sch}(\mathcal{G}_{\text{ét}}(\mathbb{Z}))$ of $\mathcal{G}_{\text{ét}}(\mathbb{Z})$-schemes in the sense of [31, 2.3.9. 2.6.11]. The constructions of this section can be extended to $\mathrm{Sch}(\mathcal{G}_{\text{ét}}(\mathbb{Z}))$ by hyperdescent. We will provide more details in [28].

6. Summary and complements

In this chapter we summarize the construction in the previous chapter and presents several complements. In §6.1, we write down the resulting six operations for the most general situations and summarize their properties. In §6.2, we develop a theory of constructible complexes, based on finiteness results of Deligne [3, Th. finitude] and Gabber [33]. In §6.3, we show that our results for constructible complexes are completely compatible with those of Laszlo–Olsson [24]. In §6.4, we prove some adjointness properties in the Noetherian case.

6.1. **Recapitulation.** Now we can summarize our construction of Grothendieck's six operations. Let $f\colon \mathcal{Y} \to \mathcal{X}$ be a morphisms of $\mathcal{C}\mathrm{hp}^{\mathrm{Ar}}$ (resp. $\mathcal{C}\mathrm{hp}^{\mathrm{DM}}$, resp. $\mathcal{C}\mathrm{hp}^{\mathrm{DM}}$) and $\lambda = (\Xi, \Lambda)$ be an object of $\mathcal{R}\mathrm{ind}$. From ${}_{\mathcal{C}\mathrm{hp}^{\mathrm{Ar}}_{\mathsf{L}}}\mathrm{EO}$ (resp. ${}_{\mathcal{C}\mathrm{hp}^{\mathrm{DM}}}\mathrm{EO}$, resp. ${}_{\mathcal{C}\mathrm{hp}^{\mathrm{DM}}}^{\mathrm{lqf}}\mathrm{EO}$) and ${}_{\mathcal{C}\mathrm{hp}^{\mathrm{Ar}}}\mathrm{EO}^*_{\otimes}$ (resp. ${}_{\mathcal{C}\mathrm{hp}^{\mathrm{DM}}}\mathrm{EO}^*_{\otimes}$), we directly obtain three operations:

1L: $f^*\colon \mathcal{D}(\mathcal{X}, \lambda)^{\otimes} \to \mathcal{D}(\mathcal{Y}, \lambda)^{\otimes}$;

2L: $f_!\colon \mathcal{D}(\mathcal{Y}, \lambda) \to \mathcal{D}(\mathcal{X}, \lambda)$ if f is locally of finite type, λ is in $\mathcal{R}\mathrm{ind}_{\mathsf{L}\text{-tor}}$ and $\mathcal{X}$ is L-coprime (resp. f is locally of finite type and λ is in $\mathcal{R}\mathrm{ind}_{\mathrm{tor}}$, resp. f is locally quasi-finite and λ is in $\mathcal{R}\mathrm{ind}$);

3L: $-\otimes - = - \otimes_{\mathcal{X}} -\colon \mathcal{D}(\mathcal{X}, \lambda) \times \mathcal{D}(\mathcal{X}, \lambda) \to \mathcal{D}(\mathcal{X}, \lambda)$.

If $\mathcal{X}$ is a 1-Artin stack (resp. 1-DM stack), then $\mathcal{D}(\mathcal{X}, \lambda)^{\otimes}$ is equivalent to $\mathcal{D}_{\mathrm{cart}}(\mathcal{X}^{\Xi}_{\text{lis-ét}}, \Lambda)^{\otimes}$ (resp. $\mathcal{D}(\mathcal{X}^{\Xi}_{\text{ét}}, \Lambda)^{\otimes}$).

Taking right adjoints for (1L) (for the underlying ∞-category) and (2L), we obtain:

1R: $f_*\colon \mathcal{D}(\mathcal{Y}, \lambda) \to \mathcal{D}(\mathcal{X}, \lambda)$;

2R: $f^!\colon \mathcal{D}(\mathcal{X}, \lambda) \to \mathcal{D}(\mathcal{Y}, \lambda)$ under the same restriction as [2L].

For (3L), moving the first factor of the source $\mathcal{D}(\mathcal{X}, \lambda) \times \mathcal{D}(\mathcal{X}, \lambda)$ to the target side, we can write the functor $-\otimes-$ in the form $\mathcal{D}(\mathcal{X}, \lambda) \to \mathrm{Fun}^{\mathrm{L}}(\mathcal{D}(\mathcal{X}, \lambda), \mathcal{D}(\mathcal{X}, \lambda))$, because the tensor product on $\mathcal{D}(\mathcal{X}, \lambda)$ is closed. Taking opposites and applying [29, 5.2.6.2], we obtain a functor $\mathcal{D}(\mathcal{X}, \lambda)^{op} \to \mathrm{Fun}^{\mathrm{R}}(\mathcal{D}(\mathcal{X}, \lambda), \mathcal{D}(\mathcal{X}, \lambda))$, which can be written as

3R: $\mathbf{Hom}(-,-) = \mathbf{Hom}_{\mathcal{X}}(-,-)\colon \mathcal{D}(\mathcal{X}, \lambda)^{op} \times \mathcal{D}(\mathcal{X}, \lambda) \to \mathcal{D}(\mathcal{X}, \lambda)$.

Besides these six operations, for every morphism $g\colon \lambda' \to \lambda$ of $\mathcal{R}\mathrm{ind}$, we have the following monoidal functor of *extension of scalars*:

$$ \mathrm{E}^{\otimes}_g\colon \mathcal{D}(\mathcal{X}, \lambda)^{\otimes} \to \mathcal{D}(\mathcal{X}, \lambda')^{\otimes}. $$

By construction, up to equivalences, E_g commutes with f^* and E_g commutes with $f_!$ when the latter is defined.

The following proposition is a direct consequence of the map ${}_{\mathcal{C}\mathrm{hp}_{\mathsf{L}}^{\mathrm{Ar}}}\mathrm{EO}_!^*$ (resp. ${}_{\mathcal{C}\mathrm{hp}^{\mathrm{DM}}}\mathrm{EO}_!^*$).

Proposition 6.1.1 (Base Change)**.** *Let*

$$\begin{tikzcd} \mathcal{W} \arrow[r, "g"] \arrow[d, "q"'] & \mathcal{Z} \arrow[d, "p"] \\ \mathcal{Y} \arrow[r, "f"] & \mathcal{X} \end{tikzcd}$$

be a Cartesian diagram in $\mathcal{C}\mathrm{hp}_{\mathsf{L}}^{\mathrm{Ar}}$ *(resp.* $\mathcal{C}\mathrm{hp}^{\mathrm{DM}}$*, resp.* $\mathcal{C}\mathrm{hp}^{\mathrm{DM}}$*) where* p *is locally of finite type (resp. locally of finite type, resp. locally quasi-finite). Then for every object* λ *of* $\mathcal{R}\mathrm{ind}_{\mathsf{L}\text{-tor}}$ *(resp.* $\mathcal{R}\mathrm{ind}_{\mathrm{tor}}$*, resp.* $\mathcal{R}\mathrm{ind}$*), the following square*

$$\begin{tikzcd} \mathcal{D}(\mathcal{W},\lambda) \arrow[d, "q_!"'] & \mathcal{D}(\mathcal{Z},\lambda) \arrow[l, "g^*"'] \arrow[d, "p_!"] \\ \mathcal{D}(\mathcal{Y},\lambda) & \mathcal{D}(\mathcal{X},\lambda) \arrow[l, "f^*"'] \end{tikzcd}$$

is commutative up to equivalence.

In the following, we will often treat the cases of ${}_{\mathcal{C}\mathrm{hp}_{\mathsf{L}}^{\mathrm{Ar}}}\mathrm{EO}$ and ${}_{\mathcal{C}\mathrm{hp}^{\mathrm{DM}}}\mathrm{EO}$ and leave the case of ${}_{\mathcal{C}\mathrm{hp}^{\mathrm{DM}}}^{\mathrm{lqf}}\mathrm{EO}$ to the reader.

Proposition 6.1.2 (Projection Formula)**.** *Let* $f\colon \mathcal{Y} \to \mathcal{X}$ *be a morphism locally of finite type of* $\mathcal{C}\mathrm{hp}_{\mathsf{L}}^{\mathrm{Ar}}$ *(resp.* $\mathcal{C}\mathrm{hp}^{\mathrm{DM}}$*). Then for every object* λ *of* $\mathcal{R}\mathrm{ind}_{\mathsf{L}\text{-tor}}$ *(resp.* $\mathcal{R}\mathrm{ind}_{\mathrm{tor}}$*), the following square*

$$\begin{tikzcd} \mathcal{D}(\mathcal{Y},\lambda)\times\mathcal{D}(\mathcal{X},\lambda) \arrow[r, "-\otimes_{\mathcal{Y}} f^*-"] \arrow[d, "f_!\times\mathrm{id}"'] & \mathcal{D}(\mathcal{Y},\lambda) \arrow[d, "f_!"] \\ \mathcal{D}(\mathcal{X},\lambda)\times\mathcal{D}(\mathcal{X},\lambda) \arrow[r, "-\otimes_{\mathcal{X}}-"] & \mathcal{D}(\mathcal{X},\lambda) \end{tikzcd}$$

is commutative up to equivalence.

Proof. The morphism f induces a vertical 1-cell

$$\begin{tikzcd} \mathcal{Y} \arrow[r, "f"] \arrow[d, "f"'] \arrow[dr, "f"] & \mathcal{X} \arrow[d, "\mathrm{id}_{\mathcal{X}}"] \\ \mathcal{X} \arrow[r, "\mathrm{id}_{\mathcal{X}}"] & \mathcal{X} \end{tikzcd}$$

of $\delta^*_{2,\{2\}}\mathrm{Fun}(\Delta^1,\mathcal{C}\mathrm{hp}_{\mathsf{L}}^{\mathrm{Ar}})^{\mathrm{cart}}_{F^0,A\rightarrow}$ (resp. $\delta^*_{2,\{2\}}\mathrm{Fun}(\Delta^1,\mathcal{C}\mathrm{hp}^{\mathrm{DM}})^{\mathrm{cart}}_{F^0,A\rightarrow}$). Then we only need to apply $\mathrm{G}_\zeta\circ{}_{\mathcal{C}\mathrm{hp}_{\mathsf{L}}^{\mathrm{Ar}}}\mathrm{EO}$ (resp. $\mathrm{G}_\zeta\circ{}_{\mathcal{C}\mathrm{hp}^{\mathrm{DM}}}\mathrm{EO}$). □

Proposition 6.1.3 (Künneth Formula)**.** *Let* $\Delta^1\times\Lambda^2_0\to\mathcal{C}\mathrm{hp}_{\mathsf{L}}^{\mathrm{Ar}}$ *(resp.* $\mathcal{C}\mathrm{hp}^{\mathrm{DM}}$*) be a limit diagram depicted as*

$$\begin{tikzcd} \mathcal{Y}_1 \arrow[d, "f_1"'] & \mathcal{Y} \arrow[l, "q_1"'] \arrow[r, "q_2"] \arrow[d, "f"] & \mathcal{Y}_2 \arrow[d, "f_2"] \\ \mathcal{X}_1 & \mathcal{X} \arrow[l, "p_1"'] \arrow[r, "p_2"] & \mathcal{X}_2, \end{tikzcd}$$

such that f_1 and f_2 are locally of finite type. Then for every object λ of $\mathcal{R}\mathrm{ind}_{\mathsf{L}\text{-tor}}$ (resp. $\mathcal{R}\mathrm{ind}_{\mathrm{tor}}$), the following square

$$\begin{tikzcd}
\mathcal{D}(\mathcal{Y}_1,\lambda)\times\mathcal{D}(\mathcal{Y}_2,\lambda) \arrow[r, "q_1^*-\otimes_{\mathcal{Y}} q_2^*-"] \arrow[d, "f_{1!}\times f_{2!}"'] & \mathcal{D}(\mathcal{Y},\lambda) \arrow[d, "f_!"] \\
\mathcal{D}(\mathcal{X}_1,\lambda)\times\mathcal{D}(\mathcal{X}_2,\lambda) \arrow[r, "p_1^*-\otimes_{\mathcal{X}} p_2^*-"] & \mathcal{D}(\mathcal{X},\lambda)
\end{tikzcd}$$

is commutative up to equivalence.

This is a formal consequence of Base Change and Projection Formula. We include a proof here for the convenience of the reader.

Proof. The diagram of stacks can be decomposed into a diagram $\Delta^1\times\Delta^2\coprod_{\Delta^1\times\Delta^1}\Delta^2\times\Delta^1\to\mathcal{C}\mathrm{hp}_{\mathsf{L}}^{\mathrm{Ar}}$ (resp. $\mathcal{C}\mathrm{hp}^{\mathrm{DM}}$) as

$$\begin{tikzcd}
 & & \mathcal{Y} \arrow[ddll, bend right, "q_1"'] \arrow[dl, "\tilde{q}_1"'] \arrow[dd, "f"] \arrow[dr, "\tilde{q}_2"] \arrow[ddrr, bend left, "q_2"] & & \\
 & \mathcal{Y}_1\times_{\mathcal{X}_1}\mathcal{X} \arrow[dl, "\tilde{p}_1"'] \arrow[dr, "\tilde{f}_1"] & & \mathcal{X}\times_{\mathcal{X}_2}\mathcal{Y}_2 \arrow[dl, "\tilde{f}_2"'] \arrow[dr, "\tilde{p}_2"] & \\
\mathcal{Y}_1 \arrow[dr, "f_1"'] & & \mathcal{X} \arrow[dl, "p_1"] \arrow[dr, "p_2"'] & & \mathcal{Y}_2 \arrow[dl, "f_2"] \\
 & \mathcal{X}_1 & & \mathcal{X}_2, &
\end{tikzcd}$$

where the three rhombi are all Cartesian diagrams. Then we have a sequence of equivalences of functors:

$$\begin{aligned}
f_!\left((q_1^*-)\otimes(q_2^*-)\right) &\simeq \tilde{f}_{2!}\tilde{q}_{2!}\left((\tilde{q}_1^*\tilde{p}_1^*-)\otimes(\tilde{q}_2^*\tilde{p}_2^*-)\right) & & \\
&\simeq \tilde{f}_{2!}\left((\tilde{q}_{2!}\tilde{q}_1^*\tilde{p}_1^*-)\otimes(\tilde{p}_2^*-)\right) & &\text{by Projection Formula}\\
&\simeq \tilde{f}_{2!}\left((\tilde{f}_2^*\tilde{f}_{1!}\tilde{p}_1^*-)\otimes(\tilde{p}_2^*-)\right) & &\text{by Base Change}\\
&\simeq (\tilde{f}_{1!}\tilde{p}_1^*-)\otimes(\tilde{f}_{2!}\tilde{p}_2^*-) & &\text{by Projection Formula}\\
&\simeq (p_1^*f_{1!}-)\otimes(p_2^*f_{2!}-) & &\text{by Base Change.}
\end{aligned}$$

□

Proposition 6.1.4. *Let $f\colon\mathcal{Y}\to\mathcal{X}$ be a morphism of $\mathcal{C}\mathrm{hp}_{\mathsf{L}}^{\mathrm{Ar}}$ (resp. $\mathcal{C}\mathrm{hp}^{\mathrm{DM}}$), and λ be an object of $\mathcal{R}\mathrm{ind}$. Then*

(1) *The functors $f^*(-\otimes-)$ and $(f^*-)\otimes(f^*-)$ are equivalent.*
(2) *The functors $\mathbf{Hom}_{\mathcal{X}}(-,f_*-)$ and $f_*\mathbf{Hom}_{\mathcal{Y}}(f^*-,-)$ are equivalent.*
(3) *Assume that f is locally of finite type; λ is in $\mathcal{R}\mathrm{ind}_{\mathsf{L}\text{-tor}}$ and $\mathcal{X}$ is L-coprime (resp. λ is in $\mathcal{R}\mathrm{ind}_{\mathrm{tor}}$). The functors $f^!\mathbf{Hom}_{\mathcal{X}}(-,-)$ and $\mathbf{Hom}_{\mathcal{Y}}(f^*-,f^!-)$ are equivalent.*
(4) *Assume that f is locally of finite type; λ is in $\mathcal{R}\mathrm{ind}_{\mathsf{L}\text{-tor}}$ and $\mathcal{X}$ is L-coprime (resp. λ is in $\mathcal{R}\mathrm{ind}_{\mathrm{tor}}$). The functors $f_*\mathbf{Hom}_{\mathcal{Y}}(-,f^!-)$ and $\mathbf{Hom}_{\mathcal{X}}(f_!-,-)$ are equivalent.*

Proof. (1) This follows from the fact that f^* is a symmetric monoidal functor.

(2) The functor $\mathbf{Hom}(-,f_*-)\colon\mathcal{D}(\mathcal{X},\lambda)^{op}\times\mathcal{D}(\mathcal{Y},\lambda)\to\mathcal{D}(\mathcal{X},\lambda)$ induces a functor $\mathcal{D}(\mathcal{X},\lambda)^{op}\to\mathrm{Fun}^{\mathrm{R}}(\mathcal{D}(\mathcal{Y},\lambda),\mathcal{D}(\mathcal{X},\lambda))$. Taking opposite, we obtain a functor $\mathcal{D}(\mathcal{X},\lambda)\to\mathrm{Fun}^{\mathrm{L}}(\mathcal{D}(\mathcal{X},\lambda),\mathcal{D}(\mathcal{Y},\lambda))$, which induces a functor $\mathcal{D}(\mathcal{X},\lambda)\times\mathcal{D}(\mathcal{X},\lambda)\to\mathcal{D}(\mathcal{Y},\lambda)$. By construction, the latter is equivalent to the functor $f^*(-\otimes-)$. Repeating the same process for $f_*\mathbf{Hom}(f^*-,-)$, we obtain $(f^*-)\otimes(f^*-)$. Therefore, by (1), $\mathbf{Hom}(-,f_*-)$ and $f_*\mathbf{Hom}(f^*-,-)$ are equivalent.
(3) The functor $f^!\mathbf{Hom}(-,-)\colon\mathcal{D}(\mathcal{X},\lambda)^{op}\times\mathcal{D}(\mathcal{X},\lambda)\to\mathcal{D}(\mathcal{Y},\lambda)$ induces a functor. $\mathcal{D}(\mathcal{X},\lambda)^{op}\to\mathrm{Fun}^{\mathrm{R}}(\mathcal{D}(\mathcal{X},\lambda),\mathcal{D}(\mathcal{Y},\lambda))$. Taking opposite, we obtain a functor $\mathcal{D}(\mathcal{X},\lambda)\to\mathrm{Fun}^{\mathrm{L}}(\mathcal{D}(\mathcal{Y},\lambda),\mathcal{D}(\mathcal{X},\lambda))$, which induces a functor $\mathcal{D}(\mathcal{X},\lambda)\times\mathcal{D}(\mathcal{Y},\lambda)\to\mathcal{D}(\mathcal{X},\lambda)$. By construction, the latter is equivalent to

the functor $-\otimes(f_!-)\colon$. Repeating the same process for $\mathbf{Hom}(f^*-,f^!-)$, we obtain $f_!((f^*-)\otimes-)$. Therefore, by Proposition 6.1.2, $f^!\mathbf{Hom}(-,-)$ and $\mathbf{Hom}(f^*-,f^!-)$ are equivalent.

(4) The functor $f_*\mathbf{Hom}(-,f^!-)\colon \mathcal{D}(\mathcal{Y},\lambda)^{op}\times\mathcal{D}(\mathcal{X},\lambda)\to\mathcal{D}(\mathcal{X},\lambda)$ induces a functor $\mathcal{D}(\mathcal{Y},\lambda)^{op}\to \mathrm{Fun}^{\mathrm{R}}(\mathcal{D}(\mathcal{X},\lambda),\mathcal{D}(\mathcal{X},\lambda))$. Taking opposite, we obtain a functor $\mathcal{D}(\mathcal{Y},\lambda)\to\mathrm{Fun}^{\mathrm{L}}(\mathcal{D}(\mathcal{X},\lambda),\mathcal{D}(\mathcal{X},\lambda))$, which induces a functor $\mathcal{D}(\mathcal{Y},\lambda)\times\mathcal{D}(\mathcal{X},\lambda)\to\mathcal{D}(\mathcal{X},\lambda)$. By construction, the latter is equivalent to the functor $f_!(-\otimes(f^*-))$. Repeating the same process for $\mathbf{Hom}(f_!-,-)$, we obtain $(f_!-)\otimes-$. Therefore, by Proposition 6.1.2, $f_*\mathbf{Hom}(-,f^!-)$ and $\mathbf{Hom}(f_!-,-)$ are equivalent.

□

For an object $\lambda=(\Xi,\Lambda)$ of $\mathcal{R}\mathrm{ind}$ and $\mathcal{X}$ of $\mathcal{C}\mathrm{hp}^{\mathrm{Ar}}$, there is a t-structure on $\mathcal{D}(\mathcal{X},\lambda)$. If $\mathcal{X}$ is a 1-Artin stack (resp. 1-DM stack), this t-structure induces the usual t-structure on its homotopy category $\mathrm{D}_{\mathrm{cart}}(\mathcal{X}^{\Xi}_{\text{lis-ét}},\Lambda)$ (resp. $\mathrm{D}(\mathcal{X}^{\Xi}_{\text{ét}},\Lambda)$). In particular, the heart $\mathcal{D}^{\heartsuit}(\mathcal{X},\lambda)$ is canonically equivalent to (the nerve of) the abelian category $\mathrm{Mod}_{\mathrm{cart}}(\mathcal{X}^{\Xi}_{\text{lis-ét}},\Lambda)$ (resp. $\mathrm{Mod}(\mathcal{X}^{\Xi}_{\text{ét}},\Lambda)$).

For an object $s\colon\mathcal{X}\to\operatorname{Spec}\mathbb{Z}$ of $\mathcal{C}\mathrm{hp}^{\mathrm{Ar}}$, we let $\Lambda_{\mathcal{X}}=s^*\Lambda_{\operatorname{Spec}\mathbb{Z}}$ be a monoidal unit, which is an object of $\mathcal{D}^{\heartsuit}(\mathcal{X},\lambda)\subseteq\mathcal{D}(\mathcal{X},\lambda)$. We have the following.

Proposition 6.1.5 (Poincaré duality)**.** *Let $f\colon\mathcal{Y}\to\mathcal{X}$ be a flat (resp. flat and locally quasi-finite) morphism of $\mathcal{C}\mathrm{hp}^{\mathrm{Ar}}_{\mathsf{L}}$ (resp. $\mathcal{C}\mathrm{hp}^{\mathrm{DM}}$), locally of finite presentation. Let λ be an object of $\mathcal{R}\mathrm{ind}_{\mathsf{L}\text{-tor}}$ (resp. $\mathcal{R}\mathrm{ind}$). Then*

(1) *There is a trace map $\mathrm{Tr}_f\colon\tau^{\geq 0}f_!\lambda_{\mathcal{Y}}\langle d\rangle=\tau^{\geq 0}f_!(f^*\lambda_{\mathcal{X}})\langle d\rangle\to\lambda_{\mathcal{X}}$ for every integer $d\geq\dim^+(f)$, which is functorial in the sense of Remark 3.3.2.*

(2) *If f is moreover smooth, the induced natural transformation $u_f\colon f_!\circ f^*\langle\dim f\rangle\to\mathrm{id}_{\mathcal{X}}$ is a counit transformation, so that the induced map $f^*\langle\dim f\rangle\to f^!$ is a natural equivalence of functors $\mathcal{D}(\mathcal{X},\lambda)\to\mathcal{D}(\mathcal{Y},\lambda)$.*

Combining Base Change and Proposition 6.1.5 (2), we obtain the following.

Corollary 6.1.6 (Smooth (resp. Étale) Base Change)**.** *Let*

$$\begin{tikzcd} \mathcal{W} \arrow[r, "g"] \arrow[d, "q"'] & \mathcal{Z} \arrow[d, "p"] \\ \mathcal{Y} \arrow[r, "f"] & \mathcal{X} \end{tikzcd}$$

be a Cartesian diagram in $\mathcal{C}\mathrm{hp}^{\mathrm{Ar}}_{\mathsf{L}}$ (resp. $\mathcal{C}\mathrm{hp}^{\mathrm{DM}}$) where p is smooth (resp. étale). Then for every object λ of $\mathcal{R}\mathrm{ind}_{\mathsf{L}\text{-tor}}$ (resp. $\mathcal{R}\mathrm{ind}$), the following square

$$\begin{tikzcd} \mathcal{D}(\mathcal{W},\lambda) & \mathcal{D}(\mathcal{Z},\lambda) \arrow[l, "g^*"'] \\ \mathcal{D}(\mathcal{Y},\lambda) \arrow[u, "q^*"] & \mathcal{D}(\mathcal{X},\lambda) \arrow[l, "f^*"] \arrow[u, "p^*"'] \end{tikzcd}$$

is right adjointable.

Proposition 6.1.7. *Let $f\colon\mathcal{Y}\to\mathcal{X}$ be a morphism of $\mathcal{C}\mathrm{hp}^{\mathrm{Ar}}_{\mathsf{L}}$ (resp. $\mathcal{C}\mathrm{hp}^{\mathrm{DM}}$), λ be an object of $\mathcal{R}\mathrm{ind}_{\mathsf{L}\text{-tor}}$ (resp. $\mathcal{R}\mathrm{ind}_{\mathrm{tor}}$). Assume that for every morphism $X\to\mathcal{X}$ from an algebraic space, the base change $\mathcal{Y}\times_{\mathcal{X}}X\to X$ is a proper morphism of algebraic spaces, which implies that f is locally of finite type. Then f_* and $f_!$ are equivalent functors $\mathcal{D}(\mathcal{Y},\lambda)\to\mathcal{D}(X,\lambda)$.*

Proof. Let us first show that f_* satisfies base change by any morphism $g\colon\mathcal{Z}\to\mathcal{X}$. For this, choose a commutative diagram

$$\begin{tikzcd} Z \arrow[r, "h"] \arrow[d, "z"'] & X \arrow[d, "x"] \\ \mathcal{Z} \arrow[r, "g"] & \mathcal{X} \end{tikzcd}$$

where X and Z are algebraic spaces, x and z are atlases. Since z^* is conservative and base change by x and z holds, we are reduced to show that base change by h, which follows from Proposition 5.2.4. The equivalence of f_* and $f_!$ then follows from Proposition 5.2.4 and recursive applications of 4.1.1. $\square$

By construction, a smooth surjective morphism of higher Artin stacks is of ${}_{\mathcal{C}\mathrm{hp}^{\mathrm{Ar}}}\mathrm{EO}^*_{\otimes}$-descent. And a smooth surjective morphism of L-coprime higher Artin stacks (resp. higher DM stacks) is of ${}_{\mathcal{C}\mathrm{hp}^{\mathrm{Ar}}_{\mathsf{L}}}\mathrm{EO}_!$-codescent (resp. ${}_{\mathcal{C}\mathrm{hp}^{\mathrm{DM}}}\mathrm{EO}_!$-codescent). In other words, we have the following proposition, which implies Theorem 0.1.8 by [30, 1.2.4.7] and its dual version.

Proposition 6.1.8 ((Co)homological descent)**.** *Let $f\colon X_0^+ \to X_{-1}^+$ be a smooth surjective morphism in $\mathcal{C}\mathrm{hp}^{\mathrm{Ar}}$ (resp. $\mathcal{C}\mathrm{hp}^{\mathrm{DM}}$) and let $X_\bullet^+$ be a Čech nerve of f.*

1. *For any object λ of $\mathcal{R}\mathrm{ind}$, the map $\mathcal{D}(X_{-1}^+,\lambda) \to \varprojlim_{n\in\mathbf{\Delta}} \mathcal{D}(X_n^+,\lambda)$ is an equivalence, where the transition maps in the limit are provided by $*$-pullback.*
2. *For any object λ of $\mathcal{R}\mathrm{ind}_{\mathsf{L}\text{-tor}}$ (resp. $\mathcal{R}\mathrm{ind}_{\mathrm{tor}}$) and when X_{-1}^+ is in $\mathcal{C}\mathrm{hp}^{\mathrm{Ar}}_{\mathsf{L}}$ (resp. $\mathcal{C}\mathrm{hp}^{\mathrm{DM}}$), the map $\varinjlim_{n\in\mathbf{\Delta}} \mathcal{D}(X_n^+,\lambda) \to \mathcal{D}(X_{-1}^+,\lambda)$ is an equivalence, where the transition maps in the colimit are provided by $!$-pushforward.*

The following lemma will be used in §6.2.

Lemma 6.1.9. *Let $f\colon Y \to X$ be a morphism locally of finite type of $\mathcal{C}\mathrm{hp}^{\mathrm{Ar}}_{\mathsf{L}}$ (resp. $\mathcal{C}\mathrm{hp}^{\mathrm{DM}}$), λ be an object of $\mathcal{R}\mathrm{ind}_{\mathsf{L}\text{-tor}}$ (resp. $\mathcal{R}\mathrm{ind}_{\mathrm{tor}}$). Then $f_!$ induces $\mathcal{D}^{\le 0}(Y,\lambda) \to \mathcal{D}^{\le 2d}(X,\lambda)$, where $d = \dim^+(f)$. Moreover, if f is smooth (resp. étale), then $f_! \circ f^!\colon \mathcal{D}(X,\lambda) \to \mathcal{D}(X,\lambda)$ is left t-exact.*

Proof. We may assume that X is the spectrum of a separably closed field.

We prove the first assertion by induction on k when Y is a k-Artin stack. Let $\mathscr{K} \in \mathcal{D}^{\le 0}(Y,\lambda)$. For $k=-2$, Y is the coproduct of a family $(Y_i)_{i\in I}$ of morphisms of schemes separated and of finite type over X, so that $f_!\mathscr{K} = \bigoplus_{i\in I} f_{i!}(\mathscr{K} \mid Y_i) \in \mathcal{D}^{\le 2d}(X,\lambda)$, where f_i is the composite morphism $Y_i \to Y \xrightarrow{f} X$. Assume the assertion proved for some $k \ge -2$, and let Y be a $(k+1)$-Artin stack. Let $Y_\bullet$ be a Čech nerve of an atlas (resp. étale atlas) $y_0\colon Y_0 \to Y$ and form the triangle (4.15). Then, by Proposition 6.1.8 (2), $f_!\mathscr{K} \simeq \varinjlim_{n\in\mathbf{\Delta}^{op}} f_{n!}y_n^!\mathscr{K}$. Thus it suffices to show that for every smooth (resp. étale) morphism $g\colon Z \to X$ where Z is a k-Artin stack, $(f\circ g)_!g^!\mathscr{K}$ is in $\mathcal{D}^{\le 2d}(X,\lambda)$. For this, we may assume that g is of pure dimension e (resp. 0). The assertion then follows from Proposition 6.1.5 and induction hypothesis.

For the second assertion, we may assume that f is of pure dimension d (resp. 0). The second assertion then follows from Proposition 6.1.5 (2) and the first assertion. $\square$

6.2. **Constructible complexes.** We study constructible complexes on higher Artin stacks and their behavior under the six operations. Let $\lambda = (\Xi,\Lambda)$ be a *Noetherian* ringed diagram. For every object ξ of Ξ, we denote by s_ξ the morphism $(\{\xi\},\Lambda(\xi)) \to (\Xi,\Lambda)$.

Definition 6.2.1. Let X be a scheme. We say that an object $\mathscr{K}$ of $\mathcal{D}(X,\lambda)$ is *constructible* if for every object ξ of Ξ and every $q\in\mathbb{Z}$, $\mathrm{H}^q s_\xi^*\mathscr{K} \in \mathrm{Mod}(X,\Lambda)$ is constructible [2, IX 2.3]. We say that an object $\mathscr{K}$ of $\mathcal{D}(X,\lambda)$ is *locally bounded from below* (resp. *locally bounded above*) if for every object ξ of Ξ and every quasi-compact open subscheme U of X, $s_\xi^*\mathscr{K} \mid U$ is bounded below (resp. bounded above).

Let $f\colon Y \to X$ be a morphism of schemes. Then f^* preserves constructible complexes by [2, IX 2.4 (iii)]. Moreover, $\mathscr{K} \in \mathcal{D}(X,\lambda)$ is locally bounded below (resp. from above) if and only if there exists a Zariski open covering $(U_i)_{i\in I}$ of X such that $\mathscr{K} \mid U_i$ is bounded below (resp. from above). It follows that f^* preserves locally bounded complex from below (resp. from above). Therefore, Definition 6.2.1 is compatible with the following.

Definition 6.2.2 (Constructible complex). Let X be a higher Artin stack. We say an object $\mathscr{K}$ of $\mathcal{D}(X,\lambda)$ is *constructible* (resp. *locally bounded below*, resp. *locally bounded above*) if for every atlas $f\colon Y \to X$ with Y a scheme, $f^*\mathscr{K}$ is constructible (resp. locally bounded from below, resp. locally bounded above).

We denote by $\mathcal{D}_{\mathrm{cons}}(X,\lambda)$ (resp. $\mathcal{D}^{(+)}(X,\lambda)$, $\mathcal{D}^{(-)}(X,\lambda)$ or $\mathcal{D}^{(\mathrm{b})}(X,\lambda)$) the full subcategory of $\mathcal{D}(X,\lambda)$ spanned by objects that are constructible (resp. locally bounded below, locally bounded above, or locally

bounded from both sides). Moreover, we let

$$\mathcal{D}^{(+)}_{\mathrm{cons}}(X,\lambda)=\mathcal{D}_{\mathrm{cons}}(X,\lambda)\cap\mathcal{D}^{(+)}(X,\lambda);$$
$$\mathcal{D}^{(-)}_{\mathrm{cons}}(X,\lambda)=\mathcal{D}_{\mathrm{cons}}(X,\lambda)\cap\mathcal{D}^{(-)}(X,\lambda);$$
$$\mathcal{D}^{(\mathrm{b})}_{\mathrm{cons}}(X,\lambda)=\mathcal{D}_{\mathrm{cons}}(X,\lambda)\cap\mathcal{D}^{(\mathrm{b})}(X,\lambda).$$

Lemma 6.2.3.

(1) *Let $f\colon Y\to X$ be a morphism of higher Artin stacks, $\mathscr{K}$ be an object of $\mathcal{D}(X,\lambda)$. If $\mathscr{K}$ is constructible (resp. locally bounded below, resp. locally bounded above), then $f^*\mathscr{K}$ satisfies the same property. The converse holds when f is surjective and locally of finite presentation.*

(2) *Let X be a higher Artin stack. Then $-\otimes_X-$ induces*

3L': $-\otimes_X -\colon \mathcal{D}^{(-)}_{\mathrm{cons}}(X,\lambda)\times\mathcal{D}^{(-)}_{\mathrm{cons}}(X,\lambda)\to\mathcal{D}^{(-)}_{\mathrm{cons}}(X,\lambda)$.

In particular, $\mathcal{D}^{(-)}_{\mathrm{cons}}(X,\lambda)^{\otimes}$ [30, 2.2.1] is a symmetric monoidal category.

By (1), for every morphism $f\colon Y\to X$ of higher Artin stacks, f^* induces

1L': $f^*\colon\mathcal{D}_{\mathrm{cons}}(X,\lambda)\to\mathcal{D}_{\mathrm{cons}}(Y,\lambda)$.

Proof. (1) Let us show the second assertion first. Up to replacing X by an atlas, we may assume that X is a scheme. Up to replacing Y by an atlas, we may further assume that Y is a scheme. The second assertion then follows from [2, IX 2.8 (resp. 2.8.1)].

To show the first assertion, we may assume X is a scheme by the second assertion. We may further assume that Y is a scheme. In this case, the first assertion has been recalled after Definition 6.2.1.

(2) We may assume X is an affine scheme. The assertion is then trivial.

□

To state the results for the other operations, we work in a relative setting. Let S be an L-coprime higher Artin stack. Assume that there exists an atlas $S\to\mathsf{S}$, where S is either a quasi-excellent[8] scheme or a regular scheme of dimension ≤ 1. We denote by $\mathcal{C}\mathrm{hp}^{\mathrm{Ar}}_{\mathrm{lft}/\mathsf{S}}\subseteq\mathcal{C}\mathrm{hp}^{\mathrm{Ar}}_{/\mathsf{S}}$ the full subcategory spanned by morphisms $X\to\mathsf{S}$ locally of finite type.

Proposition 6.2.4. *Let $f\colon Y\to X$ be a morphism of $\mathcal{C}\mathrm{hp}^{\mathrm{Ar}}_{\mathrm{lft}/\mathsf{S}}$ and λ be a Noetherian L-torsion ringed diagram. Then the operations introduced in §6.1 restrict to the following*

2R': $f^!\colon\mathcal{D}_{\mathrm{cons}}(X,\lambda)\to\mathcal{D}_{\mathrm{cons}}(Y,\lambda)$;

3R': $\mathbf{Hom}_X(-,-)\colon\mathcal{D}^{(-)}_{\mathrm{cons}}(X,\lambda)^{op}\times\mathcal{D}^{(+)}_{\mathrm{cons}}(X,\lambda)\to\mathcal{D}^{(+)}_{\mathrm{cons}}(X,\lambda)$.

If, moreover, f is quasi-compact and quasi-separated (Definition 5.4.3), then we have

1R': $f_*\colon\mathcal{D}^{(+)}_{\mathrm{cons}}(Y,\lambda)\to\mathcal{D}^{(+)}_{\mathrm{cons}}(X,\lambda)$;

2L': $f_!\colon\mathcal{D}^{(-)}_{\mathrm{cons}}(Y,\lambda)\to\mathcal{D}^{(-)}_{\mathrm{cons}}(X,\lambda)$.

Proof. We prove by induction on k that the assertions hold when f is a morphism of k-Artin stacks. The case $k=-2$ is due to Deligne [3, Th. Finitude 1.5, 1.6] if S is regular of dimension ≤ 1 and to Gabber [33] if S is quasi-excellent. In fact, in the latter case, by arguments similar to [3, Th. Finitude 2.2], we may assume $\lambda=(*,\mathbb{Z}/n\mathbb{Z})$. Now assume that the assertions hold for some $k\ge -2$ and let f be a morphism of $(k+1)$-Artin stacks. Then (2R') follows from induction hypothesis, Proposition 6.1.5 (2) and (1L'); (3R') follows from induction hypothesis, Proposition 6.1.4 (3), Proposition 6.1.5 (2) and (1L'), (2R'). It remains to prove (1R') and (2L').

By smooth base change (Corollary 6.1.6), we may assume that X is an affine scheme. Then Y is a $(k+1)$-Artin stack, of finite type over X. It suffices to show that for every object $\mathscr{K}$ of $\mathcal{D}^{\ge 0}_{\mathrm{cons}}(Y,\lambda)$ (resp. $\mathcal{D}^{\le 0}_{\mathrm{cons}}(Y,\lambda)$), $f_*\mathscr{K}$ (resp. $f_!\mathscr{K}$) is in $\mathcal{D}^{\ge 0}_{\mathrm{cons}}(X,\lambda)$ (resp. $\mathcal{D}^{\le 2d}_{\mathrm{cons}}(Y,\lambda)$, where $d=\dim^+(f)$). Let $Y_\bullet$ be a Čech nerve of an atlas $y_0\colon Y_0\to Y$, where Y_0 is an affine scheme, and form a triangle (4.15). Then for

[8] A ring is *quasi-excellent* if it is Noetherian and satisfies conditions (2), (3) of [1, 7.8.2]. A scheme is *quasi-excellent* if it admits a Zariski open cover by spectra of quasi-excellent rings.

$n \geq 0$, f_n is a quasi-compact and quasi-separated morphism of k-Artin stacks. By Proposition 6.1.8 and (resp. the dual version of) [30, 1.2.4.7], we have a convergent spectral sequence

$$\mathrm{E}_1^{p,q} = \mathrm{H}^q(f_{p*}y_p^*\mathscr{K}) \Rightarrow \mathrm{H}^{p+q}f_*\mathscr{K}, \quad (\text{resp. } \tilde{\mathrm{E}}_1^{p,q} = \mathrm{H}^q(f_{-p!}y_{-p}^!\mathscr{K}) \Rightarrow \mathrm{H}^{p+q}f_!\mathscr{K}).$$

By induction hypothesis, $\mathrm{E}_1^{p,q}$ (resp. $\tilde{\mathrm{E}}_1^{p,q}$) is constructible for all p and q and vanishes for $p < 0$ or $q < 0$ (resp. $p > 0$ or $q > 2d$ by Lemma 6.1.9). Therefore, $f_*\mathscr{K}$ (resp. $f_!\mathscr{K}$) is in $\mathcal{D}_{\mathrm{cons}}^{\geq 0}(X, \lambda)$ (resp. $\mathcal{D}_{\mathrm{cons}}^{\leq 2d}(X, \lambda)$). □

6.3. **Compatibility with Laszlo–Olsson's work.** In this section we establish the compatibility between our theory and Laszlo–Olsson's work [24], under the (more restrictive) assumptions of the latter.

We fix $\mathsf{L} = \{\ell\}$ and a Gorenstein local ring Λ of dimension 0 and residual characteristic ℓ. We will suppress Λ from the notation when no confusion arises. Let S be an L-coprime scheme satisfying the following conditions.

(1) S is affine excellent and finite-dimensional;
(2) For every S-scheme X of finite type, there exists an étale cover $X' \to X$ such that, for every scheme Y étale and of finite type over X', $\mathrm{cd}_\ell(Y) < \infty$;
(3) S admits a global dimension function, which we fix in this section.

Remark 6.3.1. In [24, 25], the authors did not explicitly include (3) in their assumptions. However, their method relies on pinned dualizing complexes (see below), which make use of the dimension function. Careful readers may also notice that our assumption (2) is slightly weaker than the assumption on cohomological dimension in [24]; for example, (2) allows the case $\mathsf{S} = \operatorname{Spec}\mathbb{R}$ and $\ell = 2$ while the assumption in [24] does not. Nevertheless, our assumption (2) implies that the right derived functor of the countable product functor on $\mathrm{Mod}(X_{\text{ét}}, \Lambda)$ has finite cohomological dimension, which is in fact sufficient for the construction in [24].

Let $\mathcal{C}\mathrm{hp}^{\mathrm{LMB}}_{\mathrm{lft}/\mathsf{S}}$ be the full subcategory of $\mathcal{C}\mathrm{hp}^{\mathrm{Ar}}_{\mathrm{lft}/\mathsf{S}}$ spanned by (1-)Artin stacks locally of finite type over S, with quasi-compact and separated diagonal. Stacks with such diagonal are called algebraic stacks in [26] and [24, 25]. We adopt the notations $\mathrm{D}_{\mathrm{cons}}(\mathcal{X}_{\text{lis-ét}}) \subseteq \mathrm{D}_{\mathrm{cart}}(\mathcal{X}_{\text{lis-ét}})$ from §0.1. For a morphism $f\colon \mathcal{Y} \to \mathcal{X}$ of finite type (in $\mathcal{C}\mathrm{hp}^{\mathrm{LMB}}_{\mathrm{lft}/\mathsf{S}}$), Laszlo–Olsson defined functors

$$\mathrm{R}f_*\colon \mathrm{D}^{(+)}_{\mathrm{cons}}(\mathcal{Y}_{\text{lis-ét}}) \to \mathrm{D}^{(+)}_{\mathrm{cons}}(\mathcal{X}_{\text{lis-ét}}), \qquad \mathrm{R}f_!\colon \mathrm{D}^{(-)}_{\mathrm{cons}}(\mathcal{Y}_{\text{lis-ét}}) \to \mathrm{D}^{(-)}_{\mathrm{cons}}(\mathcal{X}_{\text{lis-ét}}),$$
$$\mathrm{L}f^*\colon \mathrm{D}_{\mathrm{cons}}(\mathcal{X}_{\text{lis-ét}}) \to \mathrm{D}_{\mathrm{cons}}(\mathcal{Y}_{\text{lis-ét}}), \qquad \mathrm{R}f^!\colon \mathrm{D}_{\mathrm{cons}}(\mathcal{X}) \to \mathrm{D}_{\mathrm{cons}}(\mathcal{Y}_{\text{lis-ét}}),$$
$$\mathrm{R}\mathbf{Hom}_{\mathcal{X}}\colon \mathrm{D}^{(-)}_{\mathrm{cons}}(\mathcal{X}_{\text{lis-ét}})^{op} \times \mathrm{D}^{(+)}_{\mathrm{cons}}(\mathcal{X}_{\text{lis-ét}}) \to \mathrm{D}^{(+)}_{\mathrm{cons}}(\mathcal{X}_{\text{lis-ét}}),$$
$$- \overset{\mathrm{L}}{\otimes}_{\mathcal{X}} -\colon \mathrm{D}^{(-)}_{\mathrm{cons}}(\mathcal{X}_{\text{lis-ét}}) \times \mathrm{D}^{(-)}_{\mathrm{cons}}(\mathcal{X}_{\text{lis-ét}}) \to \mathrm{D}^{(-)}_{\mathrm{cons}}(\mathcal{X}_{\text{lis-ét}}).$$

Using the methods of [24], four of the six functors can be extended to $\mathrm{D}_{\mathrm{cart}}$:

$$\mathrm{R}f_*\colon \mathrm{D}^{(+)}_{\mathrm{cart}}(\mathcal{Y}_{\text{lis-ét}}) \to \mathrm{D}^{(+)}_{\mathrm{cart}}(\mathcal{X}_{\text{lis-ét}}),$$
$$\mathrm{L}f^*\colon \mathrm{D}_{\mathrm{cart}}(\mathcal{X}_{\text{lis-ét}}) \to \mathrm{D}_{\mathrm{cart}}(\mathcal{Y}_{\text{lis-ét}}),$$
$$\mathrm{R}\mathbf{Hom}_{\mathcal{X}}\colon \mathrm{D}_{\mathrm{cart}}(\mathcal{X}_{\text{lis-ét}})^{op} \times \mathrm{D}_{\mathrm{cart}}(\mathcal{X}_{\text{lis-ét}}) \to \mathrm{D}_{\mathrm{cart}}(\mathcal{X}_{\text{lis-ét}}),$$
$$- \overset{\mathrm{L}}{\otimes}_{\mathcal{X}} -\colon \mathrm{D}_{\mathrm{cart}}(\mathcal{X}_{\text{lis-ét}}) \times \mathrm{D}_{\mathrm{cart}}(\mathcal{X}_{\text{lis-ét}}) \to \mathrm{D}_{\mathrm{cart}}(\mathcal{X}_{\text{lis-ét}}).$$

For $\mathrm{L}f^*$, one imitates [24, 4.3] and apply [24, 2.2.3] to $\mathrm{D}_{\mathrm{cart}}$. The other three functors, $\mathrm{R}f_*$, $\mathrm{R}\mathbf{Hom}_{\mathcal{X}}$, and $- \overset{\mathrm{L}}{\otimes}_{\mathcal{X}} -$, are standard functors for the lisse-étale topoi (see also Remark 6.3.3 (1)). The six operations satisfy all the usual adjointness properties. On the other hand, restricting our constructions in the two previous sections, we have

$$f_*\colon \mathcal{D}^{(+)}(\mathcal{Y}) \to \mathcal{D}^{(+)}(\mathcal{X}), \qquad f_!\colon \mathcal{D}^{(-)}_{\mathrm{cons}}(\mathcal{Y}) \to \mathcal{D}^{(-)}_{\mathrm{cons}}(\mathcal{X}),$$
$$f^*\colon \mathcal{D}(\mathcal{X}) \to \mathcal{D}(\mathcal{Y}), \qquad f^!\colon \mathcal{D}_{\mathrm{cons}}(\mathcal{X}) \to \mathcal{D}_{\mathrm{cons}}(\mathcal{Y}),$$
$$\mathbf{Hom}_{\mathcal{X}}\colon \mathcal{D}(\mathcal{X})^{op} \times \mathcal{D}(\mathcal{X}) \to \mathcal{D}(\mathcal{X}),$$
$$- \otimes_{\mathcal{X}} -\colon \mathcal{D}(\mathcal{X}) \times \mathcal{D}(\mathcal{X}) \to \mathcal{D}(\mathcal{X}).$$

By Corollary 5.3.6, we have an equivalence of categories

$$\mathrm{h}\mathcal{D}(\mathcal{X}) \simeq \mathrm{D}_{\mathrm{cart}}(\mathcal{X}_{\text{lis-ét}}), \tag{6.1}$$

and, by restriction, an equivalence $\mathrm{h}\mathcal{D}_{\mathrm{cons}}(\mathcal{X}) \simeq \mathrm{D}_{\mathrm{cons}}(\mathcal{X}_{\text{lis-ét}})$. The main result of this section is the following.

Proposition 6.3.2. *We have equivalences of functors*

$$\mathrm{h}f_* \simeq \mathrm{R}f_*, \quad \mathrm{h}f_! \simeq \mathrm{R}f_!, \quad \mathrm{h}f^* \simeq \mathrm{L}f^*, \quad \mathrm{h}f^! \simeq \mathrm{R}f^!, \quad \mathrm{h}\mathbf{Hom}_{\mathcal{X}} \simeq \mathrm{R}\mathbf{Hom}_{\mathcal{X}}, \quad \mathrm{h}(-\otimes_{\mathcal{X}} -) \simeq (- \overset{\mathrm{L}}{\otimes}_{\mathcal{X}} -),$$

compatible with (6.1).

Proof. The assertion for $-\otimes_{\mathcal{X}}-$ follows from construction. By adjunction, the assertion for $\mathbf{Hom}_{\mathcal{X}}$ holds. Moreover, by adjunction, the assertion for f_* (resp. $f_!$) will follow from the one for f^* (resp. $f^!$).

Let us first prove that $\mathrm{h}f^* \simeq \mathrm{L}f^* \colon \mathrm{D}_{\mathrm{cart}}(\mathcal{X}_{\text{lis-ét}}) \to \mathrm{D}_{\mathrm{cart}}(\mathcal{Y}_{\text{lis-ét}})$. We choose a commutative diagram

$$\begin{tikzcd} Y \arrow[r] \arrow[d] & X \arrow[d] \\ \mathcal{Y} \arrow[r] & \mathcal{X} \end{tikzcd}$$

where the vertical morphisms are atlases. It induces a 2-commutative diagram

$$\begin{tikzcd} Y_\bullet \arrow[r, "f_\bullet"] \arrow[d, "\eta_Y"'] & X_\bullet \arrow[d, "\eta_X"] \\ \mathcal{Y} \arrow[r, "f"] & \mathcal{X}. \end{tikzcd}$$

Using arguments similar to §5.4, we get the following diagram

$$\begin{tikzcd} \mathcal{D}_{\mathrm{cart}}(\mathrm{Mod}(Y_{\bullet,\text{ét}})) \arrow[dr] & & \mathcal{D}_{\mathrm{cart}}(\mathrm{Mod}(X_{\bullet,\text{ét}})) \arrow[ll, "f^*_{\bullet,\text{ét}}"'] \arrow[dr] & \\ & \lim\limits_{\longleftarrow n\in\mathbf{\Delta}} \mathcal{D}(Y_{n,\text{ét}}) & & \lim\limits_{\longleftarrow n\in\mathbf{\Delta}} \mathcal{D}(X_{n,\text{ét}}) \arrow[ll, "\lim\limits_{\longleftarrow n\in\mathbf{\Delta}} f^*_{n,\text{ét}}"'] \\ \mathcal{D}_{\mathrm{cart}}(\mathcal{Y}_{\text{lis-ét}}) \arrow[uu, "\eta^*_{Y,\mathrm{cart}}"] \arrow[ur, "\sim"] & & \mathcal{D}_{\mathrm{cart}}(\mathcal{X}_{\text{lis-ét}}). \arrow[ll, dashed, "f^*"'] \arrow[uu, "\eta^*_{X,\mathrm{cart}}"'] \arrow[ur, "\sim"'] & \end{tikzcd}$$

By [24, 2.2.3], $\eta^*_{X,\mathrm{cart}}$ and $\eta^*_{Y,\mathrm{cart}}$ are equivalences. By the construction of $\mathrm{L}f^*$, $\mathrm{L}f^*$ fits into a homotopy version of the rectangle in the above diagram. Therefore, we have an equivalence $\mathrm{h}f^* \simeq \mathrm{L}f^*$.

Let $\Omega_{\mathsf{S}} \in \mathcal{D}(\mathsf{S})$ be a potential dualizing complex (with respect to the fixed dimension function) in the sense of [35, 2.1.2], which is unique up to isomorphism by [35, 5.1.1] (see Remark 6.3.3 (1)). For every object $\mathcal{X}$ of $\mathcal{C}\mathrm{hp}^{\mathrm{LMB}}_{\mathrm{lft}/\mathsf{S}}$, with structure morphism $a \colon \mathcal{X} \to \mathsf{S}$, we let $\Omega_{\mathcal{X}} = a^!\Omega_{\mathsf{S}}$. Let $u \colon U \to \mathcal{X}$ be an object of Lis-ét$(\mathcal{X})$. Then $u^*\Omega_{\mathcal{X}} \simeq \Omega_U\langle -d\rangle$ by Poincaré duality (Proposition 6.1.5 (2)), where $d = \dim u$. Consider the morphism of topoi $(\epsilon_*, \epsilon^*) \colon (\mathcal{X}_{\text{lis-ét}})_{/\widetilde{U}} \to U_{\text{ét}}$. Applying 5.3.2, we get an equivalence $\Omega_{\mathcal{X}} \mid (\mathcal{X}_{\text{lis-ét}})_{/\widetilde{U}} \simeq \epsilon^*\Omega_U\langle -d\rangle$, where we regard $\Omega_{\mathcal{X}}$ as an object of $\mathcal{D}_{\mathrm{cart}}(\mathcal{X}_{\text{lis-ét}})$ and Ω_U as an object of $\mathcal{D}(U_{\text{ét}})$. The equivalence is compatible with restriction by morphisms in Lis-ét$(\mathcal{X})$, so that $\Omega_{\mathcal{X}}$ is a dualizing complex of $\mathcal{X}$ in the sense of [24, 3.4.5], which is unique up to isomorphism by [24, 3.4.3, 3.4.4]. Let $\mathcal{D}_{\mathcal{X}} = \mathbf{Hom}_{\mathcal{X}}(-, \Omega_{\mathcal{X}})$, $\mathrm{D}_{\mathcal{X}} = \mathrm{R}\mathbf{Hom}_{\mathcal{X}}(-, \Omega_{\mathcal{X}}) \simeq \mathrm{h}\mathcal{D}_{\mathcal{X}}$. By [24, 3.5.7], the biduality functor $\mathrm{id} \to \mathrm{D}_{\mathcal{X}} \circ \mathrm{D}_{\mathcal{X}}$ is a natural isomorphism of endofunctors of $\mathrm{D}_{\mathrm{cons}}(\mathcal{X}_{\text{lis-ét}})$. Therefore, the natural transformation $\mathrm{h}f^! \to \mathrm{h}f^! \circ \mathrm{D}_{\mathcal{X}} \circ \mathrm{D}_{\mathcal{X}}$ is a natural equivalence when restricted to $\mathrm{D}_{\mathrm{cons}}(\mathcal{X}_{\text{lis-ét}})$. By Proposition 6.1.4 (3),

$$f^! \circ \mathcal{D}_{\mathcal{X}} \circ \mathcal{D}_{\mathcal{X}} \simeq f^!\mathbf{Hom}_{\mathcal{X}}(\mathcal{D}_{\mathcal{X}}-, \Omega_{\mathcal{X}}) \simeq \mathbf{Hom}_{\mathcal{Y}}(f^*\mathcal{D}_{\mathcal{X}}-, f^!\Omega_{\mathcal{X}}) \simeq \mathbf{Hom}_{\mathcal{Y}}(f^*\mathcal{D}_{\mathcal{X}}-, \Omega_{\mathcal{Y}}) = \mathcal{D}_{\mathcal{Y}} \circ f^* \circ \mathcal{D}_{\mathcal{X}}.$$

Since $\mathrm{h}f^* \simeq \mathrm{L}f^*$, this shows

$$\mathrm{h}f^! \simeq \mathrm{D}_{\mathcal{Y}} \circ \mathrm{L}f^* \circ \mathrm{D}_{\mathcal{X}} = \mathrm{R}f^!,$$

where the last identity is the definition of $\mathrm{R}f^!$ in [24, 4.4.1]. □

Remark 6.3.3.

(1) As Joël Riou observed (private communication), although the definition, existence and uniqueness of potential dualizing complexes are only stated for the coefficient ring $R=\mathbb{Z}/n\mathbb{Z}$ in [35, 2.1.2, 5.1.1], they can be extended to any Noetherian ring R' over R. In fact, if δ is a dimension function of an excellent $\mathbb{Z}[1/n]$-scheme X and K_R is a potential dualizing complex for (X,δ) relative to R, then $K_{R'}=R'\otimes_R K_R$ is a potential dualizing complex for (X,δ) relative to R' by a projection formula that follows from the fact that the punctured strict localizations of X have finite cohomological dimensions [19, 1.4]. Moreover, by the theorem of local biduality [35, 6.1.1, 7.1.2], $K_{R'}$ is a dualizing complex for $\mathrm{D}^{\mathrm{b}}_{\mathrm{cons}}(X_{\text{ét}},R')$ in the sense of [35, 7.1.1] as long as R' is Gorenstein of dimension 0.

(2) Let $f\colon \mathcal{Y}\to\mathcal{X}$ be a general morphism of Artin stacks, Λ be a general ring. Then we can define $\mathrm{L}f^*$ similarly to [32, (9.16.2)], so that we have functors

$$\mathrm{L}f^*\colon \mathrm{D}^+_{\mathrm{cart}}(\mathcal{X}_{\text{lis-ét}},\Lambda)\to \mathrm{D}^+_{\mathrm{cart}}(\mathcal{Y}_{\text{lis-ét}},\Lambda),$$
$$-\overset{\mathrm{L}}{\otimes}_{\mathcal{X}}-\colon \mathrm{D}_{\mathrm{cart}}(\mathcal{X}_{\text{lis-ét}},\Lambda)\times \mathrm{D}_{\mathrm{cart}}(\mathcal{X}_{\text{lis-ét}},\Lambda)\to \mathrm{D}_{\mathrm{cart}}(\mathcal{X}_{\text{lis-ét}},\Lambda).$$

If there exists L such that Λ is L-torsion and $\mathcal{X}$ is L-coprime, then the functors $\mathrm{R}f_*$ and $\mathrm{R}\mathbf{Hom}_{\mathcal{X}}$ for the lisse-étale topoi induce

$$\mathrm{R}f_*\colon \mathrm{D}^+_{\mathrm{cart}}(\mathcal{Y}_{\text{lis-ét}},\Lambda)\to \mathrm{D}^+_{\mathrm{cart}}(\mathcal{X}_{\text{lis-ét}},\Lambda),$$
$$\mathrm{R}\mathbf{Hom}_{\mathcal{X}}\colon \mathrm{D}_{\mathrm{cart}}(\mathcal{X}_{\text{lis-ét}},\Lambda)^{op}\times \mathrm{D}_{\mathrm{cart}}(\mathcal{X}_{\text{lis-ét}},\Lambda)\to \mathrm{D}_{\mathrm{cart}}(\mathcal{X}_{\text{lis-ét}},\Lambda).$$

Indeed, the statement for $\mathrm{R}f_*$, similar to [32, 9.9], follows from smooth base change, while the statement for $\mathrm{R}\mathbf{Hom}_{\mathcal{X}}$, similar to [24, 4.2.2], follows from the fact that the map $g^*\mathrm{R}\mathbf{Hom}_X(-,-)\to \mathrm{R}\mathbf{Hom}_Y(g^*-,g^*-)$ is an equivalence for every smooth morphism of L-coprime schemes $f\colon Y\to X$, which in turn follows from Poincaré duality. Similarly to Proposition 6.3.2, the four functors above are compatible with the functors f^*, $-\otimes_{\mathcal{X}}-$, f_*, $\mathbf{Hom}_{\mathcal{X}}$ we constructed.

(3) As a consequence of Corollary 5.3.6, we also have the compatibility for extension of scalars. More precisely, given a general Artin stack $\mathcal{X}$ and a general homomorphism of rings $g\colon\Lambda\to\Lambda'$, there is an equivalence between the functor

$$\mathrm{LE}_g=-\overset{\mathrm{L}}{\otimes}_{\Lambda}\Lambda'\colon \mathrm{D}_{\mathrm{cart}}(\mathcal{X}_{\text{lis-ét}},\Lambda)\to \mathrm{D}_{\mathrm{cart}}(\mathcal{X}_{\text{lis-ét}},\Lambda')$$

and hE_g, compatible with (6.1), where $\mathrm{E}_g\colon \mathcal{D}(\mathcal{X},\Lambda)\to\mathcal{D}(\mathcal{X},\Lambda')$ is the functor constructed in §6.1.

6.4. **A remark on the Noetherian case.** Recall the following result of Gabber: for every morphism $f\colon Y\to X$ of finite type between finite-dimensional Noetherian schemes, and every prime number ℓ invertible on X, the ℓ-cohomological dimension of f_* is finite [19, 1.4]. In particular, $f_*\colon\mathcal{D}(Y,\lambda)\to\mathcal{D}(X,\lambda)$ preserves small colimits and thus admits a right adjoint.

Let S be an L-coprime higher Artin stack admitting an atlas $S\to\mathsf{S}$ where S is a scheme admitting a Zariski open covering by finite-dimensional Noetherian schemes. Let $\lambda=(\Xi,\Lambda)$ be an object of $\mathcal{R}\mathrm{ind}_{\mathsf{L}\text{-tor}}$.

Proposition 6.4.1. *Let $f\colon Y\to X$ be a morphism in $\mathcal{C}\mathrm{hp}^{\mathrm{Ar}}_{\mathrm{lft}/\mathsf{S}}$. Then $f^!\colon\mathcal{D}(X,\lambda)\to\mathcal{D}(Y,\lambda)$ admits a right adjoint. Moreover, if f is 0-Artin, quasi-compact and quasi-separated, then $f_*\colon\mathcal{D}(Y,\lambda)\to\mathcal{D}(X,\lambda)$ also admits a right adjoint.*

Proof. Let $g\colon\coprod Z_i=Z\to Y$ be an atlas of Y. By Poincaré duality, $g^!$ is conservative, and $h_i^!$ exhibits $\mathcal{D}(Z,\lambda)$ as the product of $\mathcal{D}(Z_i,\lambda)$, where $h_i\colon Z_i\to Z$. Therefore, To show that $f^!$ preserves small colimits, it suffices to show that, for every i, $(f\circ g_i)^!$ preserves small colimits, where $g_i\colon Z_i\to Y$. We may thus assume that X and Y are both affine schemes. Let i be a closed embedding of Y into an affine space over X. It then suffices to show that $i^!$ preserves small colimits, which follows from the finiteness of cohomological dimension of j_*, where j is the complementary open immersion.

For the second assertion, by smooth base change, we may assume that X is an affine Noetherian scheme. By alternating Čech resolution, we may assume that Y is a scheme. The assertion in this case has been recalled above. □

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